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Stuart Raby

Supersymmetric Grand Unified Theories

From Quarks to Strings via SUSY GUTs

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I solve physics problems while I sleep, but when awake my family always comes first. This book is dedicated to my wife, Michele, and my children, Eric and Liat, and their families.

Preface

Particle physics is the study of the fundamental building blocks of nature, i.e. the particles and their interactions. We have learned much about the four known forces of nature—the strong, weak, electromagnetic and gravitational forces. This began with the work of Coulomb, Ampére and Faraday on the phenomena of electricity and magnetism, which led to the work of Maxwell in 1863 on the unified theory of electromagnetism. The discovery of the radioactive decay of the elements in 1895 led along a jagged path to an understanding of the weak and strong forces, culminating with Einstein's general theory of relativity in 1915. Along the way a quantum theory of nature was developed to understand the world of atoms and molecules, while Einstein's special theory of relativity was needed to unite Newtonian mechanics with electrodynamics. Finally, these two paradigms were combined in the very successful formalism of relativistic quantum field theory.

Beginning in 1905 with the discovery of the electron by J.J. Thomson, the particle zoo has grown dramatically. It now includes three families of quarks and leptons. The lightest up and down quarks are the constituents of protons and neutrons, while the lightest leptons include the electron and three types of neutrinos. The two additional families of quarks and charged leptons exist for no apparent reason. These particles and their forces are the ingredients of the Standard Model of particle physics which became complete with the discovery of the Higgs boson in 2012. The Standard Model incorporates the great success of quantum electrodynamics, using the paradigm of relativistic quantum field theory to describe all particles and their interactions via the strong, weak and electromagnetic forces. The Standard Model (plus Einstein's gravity) describes, with amazing accuracy, phenomena on the smallest distance scales measured in the laboratory and the largest distance scales relevant for stars, planets and galaxies. It is used to understand the universe from the time of the Big Bang until the present.

Whereas the Standard Model is now complete, it is far from a satisfactory theory of everything. We don't understand why the four forces have dramatically different strengths. We don't know why there are three families of quarks and leptons or why they have their respective masses. There is apparently an unknown dark matter and dark energy pervading the universe, and these don't fit into the Standard Model.

We don't understand why the three types of neutrinos are massive, but all so light. Finally, we don't know if the four forces of nature are all that there is or whether they are completely independent. For example, we learned in 1973 that the electromagnetic and weak forces are not independent at all, but are unified into the electroweak interactions. Perhaps all the fundamental building blocks of nature, i.e. the particles and the four forces, are unified in some way. This idea receives traction from the fact that quarks and leptons are all apparently point like fundamental particles. J. Pati and A. Salam suggested in 1973 that perhaps quarks and leptons can also be unified in some big picture.

It is this big picture which is the focus of this book. By 1976, when I received my Ph.D., it seemed that all the necessary theoretical ingredients of the Standard Model were present. It just took another 36 years of experiment to convincingly demonstrate this point. Upon receiving my Ph.D., I began considering physics beyond the Standard Model. Grand unification of the particles and forces had already been discussed by Howard Georgi and Sheldon Glashow in 1974. From their analysis and the work of H. Georgi, H. Quinn and S. Weinberg, it was clear that grand unified theories can unify quarks and leptons and also the strong, weak and electromagnetic forces. Proton decay was predicted to occur and experiments looking for proton decay were constructed.

In 1980, while at Stanford, Leonard Susskind, Savas Dimopoulos, Hans Peter Nilles and I began studying the remarkable new theoretical construct known as "supersymmetry". Savas and I constructed supersymmetric models of particle physics which attempted to explain why the weak scale is so much smaller than the Planck scale (where gravity becomes strong). Then in 1981, for a short period of time, Frank Wilczek, Savas and I overlapped at U.C. Santa Barbara. In this brief moment, we showed that supersymmetric grand unification was consistent with all known data. We predicted that the early experiments searching for proton decay might not see anything. Finally, just ten years later, in 1991 it was shown by the LEP experiments at CERN that supersymmetric grand unification was consistent with the measured strengths of the strong, weak and electromagnetic forces. However, gravity was still an outlier. In subsequent years many attempts have been made to combine all the known particles and forces into one unified theory. In my mind, this requires embedding any theory of particle physics into string theory, which successfully incorporates a quantum theory of gravity.

In these lectures, I will describe my own attempts in this direction. To be clear, this work is not done. Moreover, at the time of this writing, it is still not known experimentally whether supersymmetry is a property of nature. Nevertheless, the theory of supersymmetric grand unification is so compelling that many physicists, including me, feel that it will eventually be discovered. Let me now begin the discussion of supersymmetric grand unified theories starting with the Standard Model and ending with a string theory description of the fundamental building blocks of nature.

Columbus, OH, USA Stuart Raby

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Contents

Chapter 1
Introduction

In this course we will discuss my personal view on Supersymmetric Grand Unified Theories [SUSY GUTs]. It therefore behooves me to start the course by laying out my perspective and thereby provide an understanding of how this course will progress. The course will be divided into three sections. In the first section we will discuss SUSY GUTs in four space-time dimensions. Complete four dimensional theories exist in the literature and they make predictions for experiments. We will discuss these predictions and/or constraints coming from present day experimental results. We will then discuss some of the ugly features of these models. Of course, beauty is subjective and in this case we will define beauty as the possibility of embedding our four dimensional SUSY GUT in a theory including gravity. Since the only known consistent quantum mechanical theory of gravity is String Theory [ST], we will require that our SUSY GUT be embeddable into ST. This will provide theoretical constraints on model building which can be satisfied by defining our SUSY GUT in extra spatial dimensions. In section two of the course we will therefore spend some time discussing so-called Orbifold GUTs and how they solve some of the problems of 4D SUSY GUTs. Unfortunately, Orbifold GUTs introduce new problems which are then resolved in the third section in which we show how to embed Orbifold GUTs into the $E_8 \otimes E_8$ Heterotic String in ten space-time dimensions.

Before beginning this review on SUSY GUTs, it is probably worthwhile spending a very brief moment motivating the topic. What are the virtues of SUSY GUTs? The following is a list of all the issues that SUSY GUTs either addresses directly or provides a framework for addressing.

1. $M_Z << M_{GUT}$ "Natural," i.e. Why is the weak scale so much smaller than either the GUT or the Planck scales.
2. Explains Charge Quantization and family structure.

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3. Predicts Gauge Coupling Unification*[1]
4. Predicts Yukawa Coupling Unification
5. and with the addition of Family Symmetries it can accommodate the Hierarchy of Fermion Masses.
6. Neutrino Masses are obtained via a See-Saw scale $\sim 10^{-3}$–$10^{-2}\ M_G$.
7. The Lightest Supersymmetric Particle [LSP] is a natural Dark Matter Candidate.
8. Baryogenesis via Leptogenesis, i.e. the matter- anti-matter asymmetry of the universe is produced by first creating a lepton asymmetry.
9. SUSY Desert (no new physics between the weak and GUT scales) means that LHC experiments probe physics of order the GUT scale.
10. SUSY GUTs are natural extensions of the Standard Model.

In the following lectures we will discuss some of these issues in great detail. Let us start by defining our notation for the Standard Model.

[1] The asterisk is there since this prediction is consistent with low energy data.

Chapter 2
Brief Review of the Standard Model

2.1 Notation

We use the following notation throughout the book. The 4D metric is given by

$$g_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.1}$$

where the indices μ, ν take on values $(0, 1, 2, 3)$. The Dirac gamma matrices in the chiral basis are given by

$$\gamma^\mu = (\gamma^0, \gamma^i) = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad i = 1, 2, 3 \ \ , \gamma_5 = \begin{pmatrix} -\mathbb{I}_{2\times 2} & 0 \\ 0 & \mathbb{I}_{2\times 2} \end{pmatrix} \tag{2.2}$$

in terms of the 2×2 blocks

$$\sigma^\mu = (\mathbb{I}_{2\times 2}, \sigma^i), \ \ \bar{\sigma}^\mu = (\mathbb{I}_{2\times 2}, -\sigma^i) \tag{2.3}$$

where σ^i are the Pauli matrices with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Also

$$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu \, \bar{\sigma}^\nu - \sigma^\nu \, \bar{\sigma}^\mu)$$

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and

$$\bar{\sigma}^{\mu\nu} = \frac{1}{4}(\bar{\sigma}^{\mu}\, \sigma^{\nu} - \bar{\sigma}^{\nu}\, \sigma^{\mu})$$

$$\epsilon_{\alpha\,\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv -i\sigma_2$$

2.2 The Standard Model

Let us define the generators for the gauge group $SU(3) \otimes SU(2) \otimes U(1)_Y$ as $\mathcal{T}_A,\ A = 1, \ldots, 8$ for $SU(3)$, $T_a,\ a = 1, 2, 3$ for $SU(2)$ and the hypercharge operator Y for $U(1)_Y$. $\mathcal{T}_A$, T_a and Y are operators which take values, defined below, on Standard Model fields. We define the Standard Model in its simplest form in terms of left-handed Weyl spinors.

The gauge interactions of the quarks and leptons of the Standard Model are then completely defined in terms of their gauge quantum numbers. The quark and lepton fields for one family are given in terms of the left-handed Weyl spinors:

$$q = \begin{pmatrix} u \\ d \end{pmatrix},\ \bar{u},\ \bar{d}, \qquad l = \begin{pmatrix} \nu \\ e \end{pmatrix},\ \bar{e},\ \bar{\nu} \tag{2.4}$$

with SM charges given by

$$\begin{aligned} &\mathcal{T}_A\, q = \frac{1}{2}\lambda_A\, q,\ \mathcal{T}_A\, \bar{u} = -\frac{1}{2}\lambda_A^T\, \bar{u},\ \mathcal{T}_A\, \bar{d} = -\frac{1}{2}\lambda_A^T\, \bar{d} \\ &\mathcal{T}_A\, l = \mathcal{T}_A\, \bar{e} = \mathcal{T}_A\, \bar{\nu} = 0 \end{aligned} \tag{2.5}$$

(where $\lambda_A,\ A = 1, \cdots, 8$ are the 3×3 Gell-Mann matrices).

$$\begin{aligned} &T_a\, q = \frac{1}{2}\tau_a\, q,\ T_a\, l = \frac{1}{2}\tau_a\, l \\ &T_a\, \bar{u} = T_a\, \bar{d} = T_a\, \bar{e} = T_a\, \bar{\nu} = 0 \end{aligned} \tag{2.6}$$

(where $\tau_a,\ \ a = 1, 2, 3$ are the 2×2 Pauli matrices). We use the convention $Tr(\mathcal{T}_A\ \mathcal{T}_B) = \frac{1}{2}\delta_{AB}$ and $Tr(T_a\ T_b) = \frac{1}{2}\delta_{ab}$.

$$Y\, q = \frac{1}{3}\, q,\ Y\, \bar{u} = -\frac{4}{3}\, \bar{u},\ Y\, \bar{d} = \frac{2}{3}\, \bar{d},\ Y\, l = -l,\ Y\, \bar{e} = +2\bar{e},\ Y\, \bar{\nu} = 0. \tag{2.7}$$

With this notation, the gauge covariant derivative is given by

$$D_\mu = (\partial_\mu + i g_s \mathcal{T}_A \mathcal{G}_{\mu A} + i g T_a W_{\mu a} + i g' \frac{Y}{2} B_\mu) \tag{2.8}$$

and the electric charge operator is given by $Q = T_3 + \frac{Y}{2}$.

Let us define matrix valued gauge fields given by

$$\tilde{\mathcal{G}}_\mu = \mathcal{T}_A \mathcal{G}_{\mu A} \quad \text{and} \quad \tilde{W}_\mu = T_a W_{\mu a}. \tag{2.9}$$

Then the gauge field strengths are simply given by

$$\tilde{\mathcal{G}}_{\mu\nu} = \partial_\mu \tilde{\mathcal{G}}_\nu - \partial_\nu \tilde{\mathcal{G}}_\mu + i g_s [\tilde{\mathcal{G}}_\mu, \tilde{\mathcal{G}}_\nu] \tag{2.10}$$

and similarly for $\tilde{W}_{\mu\nu}$.

Then the gauge-fermion Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{gauge-fermion} = & [l^* \, i\bar{\sigma}_\mu D^\mu \, l + \bar{e}^* \, i\bar{\sigma}_\mu D^\mu \, \bar{e} + \bar{\nu}^* \, i\bar{\sigma}_\mu D^\mu \, \bar{\nu} \\ & + q^* \, i\bar{\sigma}_\mu D^\mu \, q + \bar{u}^* \, i\bar{\sigma}_\mu D^\mu \, \bar{u} + \bar{d}^* \, i\bar{\sigma}_\mu D^\mu \, \bar{d}] \\ & - \frac{1}{2} Tr(\tilde{\mathcal{G}}_{\mu\nu} \tilde{\mathcal{G}}^{\mu\nu}) - \frac{1}{2} Tr(\tilde{W}_{\mu\nu} \tilde{W}^{\mu\nu}) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \end{aligned} \tag{2.11}$$

where family indices are suppressed. Note, under the discrete symmetries $\{\mathbf{C}, \mathbf{P}, \mathbf{T}\}$ the Weyl spinors transform as follows (for example for electrons)

$$\mathrm{C} \qquad e \leftrightarrow \bar{e} \tag{2.12}$$

$$\mathrm{P} \qquad e_L \leftrightarrow e_R \tag{2.13}$$

$$\mathrm{T} \quad e \leftrightarrow i\,\sigma_2\, e^*; \quad \bar{e} \leftrightarrow i\,\sigma_2\, \bar{e}^*. \tag{2.14}$$

In order to make contact with phenomenology it is often useful to use Dirac four component notation. For example, the Dirac 4 component electron field, in terms of the 2 component Weyl spinors, is given by

$$\Psi_e \equiv \begin{pmatrix} e_L \\ e_R \end{pmatrix} = \begin{pmatrix} e \\ i\sigma_2 \, (\bar{e})^* \end{pmatrix}. \tag{2.15}$$

The gauge-fermion Lagrangian for one family of leptons, using Dirac notation (and ignoring the right-handed neutrinos $\bar{\nu}$), is then given by

$$\begin{aligned} \mathcal{L}_{gauge-fermion} = & \bar{\Psi}_{\nu L} i \gamma_\mu \partial^\mu \Psi_{\nu L} + \bar{\Psi}_e i \gamma_\mu \partial^\mu \Psi_e \\ & - \frac{g}{\sqrt{2}} [W_-^\mu \bar{\Psi}_e \gamma_\mu P_L \Psi_{\nu_e} + W_+^\mu \bar{\Psi}_{\nu_e} \gamma_\mu P_L \Psi_e] + e A^\mu \bar{\Psi}_e \gamma_\mu \Psi_e \end{aligned} \tag{2.16}$$

$$-\frac{g}{2\cos\theta_W} Z^\mu [\bar{\Psi}_{\nu_e}\gamma_\mu(g_V^{\nu_e} - \gamma_5 g_A^{\nu_e})\Psi_{\nu_e} + \bar{\Psi}_e\gamma_\mu(g_V^e - \gamma_5 g_A^e)\Psi_e]$$
$$-\frac{1}{4}W_{\mu\nu a}W_a^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

where[1]

$$g_V^{\nu_e} = g_A^{\nu_e} = \frac{1}{2};\ \ g_V^e = -\frac{1}{2}(1 - 4\sin^2\theta_W),\ g_A^e = -\frac{1}{2}. \tag{2.17}$$

In addition we must add the Higgs bosons. We will introduce the minimal set of Higgs doublets consistent with supersymmetry (see next lecture).

$$H_u = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix},\ \ H_d = \begin{pmatrix} \bar{h}^0 \\ \bar{h}^- \end{pmatrix} \tag{2.18}$$

satisfying

$$\begin{aligned} &\mathscr{T}_A\, H_u = \mathscr{T}_A\, H_d = 0 \\ &T_a\, H_u = \frac{1}{2}\tau_a\, H_u,\ T_a\, H_d = \frac{1}{2}\tau_a\, H_d \\ &Y\, H_u = +H_u,\ Y\, H_d = -H_d. \end{aligned} \tag{2.19}$$

$$-\mathscr{L}_{Yukawa} = \epsilon_{\alpha\beta}\ \bar{e}_i\ Y_e^{ij}\ l_j^\alpha\ H_d^\beta + \epsilon_{\alpha\beta}\ \bar{d}_i\ Y_d^{ij}\ q_j^\alpha\ H_d^\beta + \epsilon_{\alpha\beta}\ \bar{u}_i\ Y_u^{ij}\ q_j^\beta\ \ H_u^\alpha \tag{2.20}$$
$$+\epsilon_{\alpha\beta}\ \bar{\nu}_i\ Y_\nu^{ij}\ l_j^\beta\ H_u^\alpha - \frac{1}{2}M_{ij}\ \bar{\nu}_i\ \bar{\nu}_j + h.c.$$

where the implicit fermion spinor indices are anti-symmetrized, i.e. for example, $\bar{e}\ l \equiv \bar{e}^T\ (-i\sigma_2)\ l$, $i, j = 1, 2, 3$ are family indices and $\alpha, \beta = 1, 2$ are $SU(2)$ doublet indices. The Yukawa matrices, $Y_e,\ Y_d,\ Y_u,$ and Y_ν are, in general, 3×3 complex matrices. They are assumed to be determined by some fundamental theory at some scale between 1 TeV and the Planck scale. In this course we shall assume that the fundamental scale is of order a GUT scale, $\sim 10^{16}$ GeV. Note, the SM kinetic terms (including the gauge interactions) are invariant under a large global symmetry, $U(3)_q \times U(3)_{\bar{u}} \times U(3)_{\bar{d}} \times U(3)_l \times U(3)_{\bar{e}}$. In the next section, we shall use this global symmetry to diagonalize the fermion mass matrices.

Finally we add the Higgs Lagrangian given by

$$\mathscr{L}_{Higgs} = (D^\mu H_u)^*(D_\mu H_u) + (D^\mu H_d)^*(D_\mu H_d) - V(H_u, H_d) \tag{2.21}$$

[1] See problem 1.

where the Higgs potential is chosen such that the Higgs fields obtain vacuum expectation values [VEVs] which spontaneously break $SU(2) \otimes U(1)_Y$ to $U(1)_{EM}$, i.e.

$$\langle H_u \rangle = \begin{pmatrix} 0 \\ v_u \end{pmatrix}, \quad \langle H_d \rangle = \begin{pmatrix} v_d \\ 0 \end{pmatrix} \tag{2.22}$$

with $v_u = \frac{v}{\sqrt{2}} \sin\beta, \quad v_d = \frac{v}{\sqrt{2}} \cos\beta$ and $v \approx 246\,\text{GeV}$.

Note, as in the SM with one Higgs doublet, we have the relations satisfied at tree level

$$\tan\theta_W = g'/g, \quad e = g\ \sin\theta_W, \quad M_W = M_Z\ \cos\theta_W, \quad M_W = \frac{gv}{2} \quad \text{and} \quad \frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}. \tag{2.23}$$

In Eq. (2.20) we have also added a Majorana mass term for the right-handed neutrinos, or left-handed anti-neutrinos, with mass matrix $M_{ij} = M_{ji}$. This is possible since the right-handed neutrinos are "sterile" neutrinos, i.e. they do not have any SM charges.

2.3 Fermion Masses and Mixing

When the Higgs bosons obtain their vacuum expectation values, quarks and leptons get chiral symmetry breaking masses. Their mass matrices are given by

$$m_e = Y_e\ v_d,\ m_d = Y_d\ v_d,\ m_u = Y_u\ v_u \tag{2.24}$$

and the Dirac mass matrix for the neutrinos is given by

$$m_\nu = Y_\nu\ v_u. \tag{2.25}$$

The diagonal mass matrices are obtained via two unitary rotations given by

$$\begin{aligned} m_e^D &= U_{\bar{e}}^\dagger\ m_e\ U_l \equiv \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \\ m_u^D &= U_{\bar{u}}^\dagger\ m_u\ U_q \equiv \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \\ m_d^D &= U_{\bar{d}}^\dagger\ m_d\ U_d \equiv \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}. \end{aligned} \tag{2.26}$$

For neutrinos, we choose to work in a lepton flavor basis with $U_\nu \equiv U_l$.

Upon diagonalizing the fermion mass matrices, we find that there appears a CKM mixing matrix in the coupling to charged W bosons, with

$$\mathscr{L} \supset -\frac{g}{\sqrt{2}} W_+^\mu \; (u^\dagger \; \bar{\sigma}_\mu \; V_{CKM} \; d \; + \; \nu^\dagger \; \bar{\sigma}_\mu \; e) \tag{2.27}$$

where $V_{CKM} \equiv U_q^\dagger \; U_d$, i.e. the consequence of up-down symmetry breaking in the Yukawa matrices is the fundamental reason $V_{CKM} \neq I$. The nine quark and lepton masses are very much hierarchical with $u, \; d, \; e$ lighter than $c, \; s, \; \mu$ which are lighter than $t, \; b, \; \tau$. The lightest charged fermion is the electron with mass, $m_e \sim 0.5\,\text{MeV}$, and the heaviest is the top quark with mass, $m_t \sim 173\,\text{GeV}$. In addition, the CKM matrix is also hierarchical which is most apparent in the Wolfenstein form

$$V_{CKM} \sim \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\,\lambda^3\,(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\,\lambda^2 \\ A\,\lambda^3\,(1 - \rho - i\eta) & -A\,\lambda^2 & 1 \end{pmatrix} \tag{2.28}$$

with $\lambda \sim 0.22, \; A \sim 0.8, \; \sqrt{\rho^2 + \eta^2} \sim 0.4$ and η is the CP violating parameter. The CKM matrix parametrizes flavor violation in the quark sector with generally small mixing angles, i.e. $\sin(\theta_C) \sim \lambda$.

For neutrinos, on the other hand, we assume that the Majorana mass matrix has three large eigenvalues, much larger than the electroweak VEV, v. In this case we obtain three heavy Majorana neutrinos (mostly $\bar{\nu}$s) and three light Majorana neutrinos (mostly νs) with mass matrix given by

$$\tilde{m}_\nu = U_l^T \; m_\nu^T \; M^{-1} \; m_\nu \; U_l. \tag{2.29}$$

Note the neutrino mass matrix, $\tilde{m}_\nu$, is not necessarily diagonal in the flavor basis. It can however be diagonalized via a unitary transformation, U, such that

$$\tilde{m}_\nu^D = U^T \; \tilde{m}_\nu \; U \equiv \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{2.30}$$

The unitary matrix, $U \equiv U_{PMNS}$ is measured in neutrino oscillation experiments. We have flavor eigenstates given by $|\nu_\alpha >$ with $\alpha = \{e, \; \mu, \; \tau\}$ which are transformed into mass eigenstates, $|\nu_i >$ with $i = 1, 2, 3$ via

$$|\nu_\alpha >= U_{\alpha\;i}^* \; |\nu_i > . \tag{2.31}$$

In the neutrino sector three mixing angles and two mass differences have been measured. The upper bound on the sum of neutrino masses is given by cosmology to be of order 0.12 eV. Defining $\Delta m_{ij}^2 = m_i^2 - m_j^2$ we have $\Delta m_{21}^2 \sim 7 \times 10^{-5}\,\text{eV}^2$,

$\Delta m_{32}^2 \sim \pm 2 \times 10^{-3}\,\text{eV}^2$ (with the plus sign corresponding to the normal mass hierarchy and the minus sign the inverted mass hierarchy). In addition, the PMNS matrix written in the form

$$U_{PMNS} = \begin{pmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13}\, e^{-i\delta} \\ -s_{12}\, c_{23} - c_{12}\, s_{13}\, s_{23}\, e^{i\delta} & c_{12}\, c_{23} - s_{12}\, s_{13}\, s_{23}\, e^{i\delta} & c_{13}\, s_{23} \\ s_{12}\, s_{23} - c_{12}\, s_{13}\, c_{23}\, e^{i\delta} & -c_{12}\, s_{23} - s_{12}\, s_{13}\, c_{23}\, e^{i\delta} & c_{13}\, c_{23} \end{pmatrix} \tag{2.32}$$

has $\sin^2(\theta_{12}) \sim 0.3$, $\sin^2(\theta_{23}) \sim 0.45$, $\sin^2(\theta_{13}) \sim 0.024$ and the CP violating angle (which hasn't been directly measured) satisfying $\sin\delta \sim -1$ by consistency with several experiments.

The three gauge coupling constants have been measured quite accurately and at the Z mass we have $\alpha_s(M_Z) \sim 0.11$, $\alpha_{EM}^{-1}(M_Z) \sim 128$, $\sin^2(\theta_W)(M_Z) \sim 0.23$.

The Standard Model is thus defined in terms of three experimentally measured gauge couplings; 13 charged fermion masses and mixing angles; 3 neutrino masses, 3 real mixing angles and 1 CP violating phase in the leptonic sector (2 additional CP violating phases in the neutrino sector are probably unobservable); the Higgs and Z masses and a QCD θ angle which satisfies $\theta < 10^{-10}$ or a total of 28 parameters (29 including gravity, with $G_N \equiv M_{Planck}^{-2}$ and $M_{Planck} \sim 1.2 \times 10^{19}$ GeV).

Chapter 3
Minimal Supersymmetric Standard Model

3.1 Notation

In this section we refine our spinor notation and introduce the dotted and un-dotted notation for Weyl spinors. The Lorentz group $SO(1,3)$ is homomorphic to the group $SL(2,C)$ where the latter is defined by the set of complex 2×2 matrices, M with $\det M = 1$ and group product, matrix multiplication. If we take the four vector momentum, $P_\mu = (E, \ -\mathbf{p})$ and define the matrix

$$P_\mu \, \sigma^\mu = \begin{pmatrix} E - p^3 & -(p^1 - ip^2) \\ -(p^1 + ip^2) & E + p^3 \end{pmatrix}, \tag{3.1}$$

then it is easy to see that $\det(P_\mu \, \sigma^\mu) = E^2 - \mathbf{p}^2 \equiv P_\mu \, P^\mu$ is a Lorentz scalar. The general element $M \subset SL(2,C)$ has the form $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha, \ \beta, \ \gamma, \ \delta$ are complex numbers with $\alpha \, \delta - \beta \, \gamma = 1$. Thus $SL(2,C)$ is defined by a 6 dimensional group manifold, just like the Lorentz group. Moreover, it is easy to see that the $SL(2,C)$ transformation

$$P'_\mu \, \sigma^\mu = M \, P_\mu \, \sigma^\mu \, M^\dagger \tag{3.2}$$

satisfies $(P')^2 = P^2$. Thus it generates a Lorentz transformation on two component spinors, ψ_α, such that

$$\psi'_\alpha = M_\alpha{}^\beta \, \psi_\beta. \tag{3.3}$$

We can also define spinors with upper indices, ψ^α which transform by $\psi'^\alpha = \psi^\beta \, M^{-1}{}_\beta{}^\alpha$. We note that the bilinear $(\xi \, \eta) = \xi^\alpha \, \eta_\alpha$ is a Lorentz scalar.

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The group $SL(2, C)$ has a 2×2 invariant tensor given by $\epsilon_{\alpha\ \beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv -i\sigma_2$. Then ϵ can be used to raise and lower spinor indices. Define $\eta_\alpha = \epsilon_{\alpha\ \beta}\ \eta^\beta$ and $\xi^\alpha = \epsilon^{\alpha\ \beta}\ \xi_\beta$ where $\epsilon^{\alpha\ \beta} \equiv -\epsilon_{\alpha\ \beta}$ and $\epsilon_{\beta\ \alpha}\ \epsilon^{\alpha\ \gamma} = \delta_\beta{}^\gamma$. Then

$$(\xi\ \eta) = \xi^\alpha\ \eta_\alpha = \xi^\alpha\ \epsilon_{\alpha\ \beta}\ \eta^\beta. \tag{3.4}$$

There is also a conjugate spinor representation, $\bar{\psi}_{\dot{\alpha}} \equiv \psi^*_\alpha$ transforming as $\bar{\psi}'_{\dot{\alpha}} = M^*{}_{\dot{\alpha}}{}^{\dot{\beta}}\ \bar{\psi}_{\dot{\alpha}}$ and $\bar{\psi}^{\dot{\alpha}}$ transforming as $\bar{\psi}'^{\dot{\alpha}} = M^{*-1}{}^{\dot{\alpha}}{}_{\dot{\beta}}\ \bar{\psi}^{\dot{\alpha}}$. The invariant tensor for the conjugate representation is $\epsilon_{\dot{\alpha}\ \dot{\beta}} \equiv \epsilon_{\alpha\ \beta}$ and $\epsilon^{\dot{\alpha}\ \dot{\beta}} = -\epsilon_{\dot{\alpha}\ \dot{\beta}}$. Then $\bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\ \dot{\beta}}\ \bar{\psi}_{\dot{\beta}}$, $\bar{\psi}_{\dot{\beta}} = \epsilon_{\dot{\beta}\ \dot{\alpha}}\ \bar{\psi}^{\dot{\alpha}}$ and the Lorentz scalar is given by

$$(\bar{\xi}\ \bar{\eta}) = \bar{\xi}_{\dot{\alpha}}\ \bar{\eta}^{\dot{\alpha}}. \tag{3.5}$$

Note the difference between the definitions, Eqs. (3.4) and (3.5).

With this refined notation, we have

$$\sigma_\mu = \sigma_{\mu_{\alpha\ \dot{\beta}}}, \quad \bar{\sigma}_\mu = \bar{\sigma}_\mu^{\dot{\alpha}\ \beta} \tag{3.6}$$

and

$$\begin{aligned} (\sigma^{\mu\nu})_\alpha{}^\beta &= \tfrac{1}{4}(\sigma^\mu\ \bar{\sigma}^\nu - \sigma^\nu\ \bar{\sigma}^\mu)_\alpha{}^\beta \\ (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} &= \tfrac{1}{4}(\bar{\sigma}^\mu\ \sigma^\nu - \bar{\sigma}^\nu\ \sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}}. \end{aligned} \tag{3.7}$$

Finally, a Dirac electron field, Ψ_e, is given in terms of two left-handed Weyl fermion fields, $e, \bar{e}$ via

$$\Psi_e = \begin{pmatrix} e_\alpha \\ i\sigma_2\ (\bar{e}_\beta)^* \end{pmatrix} = \begin{pmatrix} e_\alpha \\ (\epsilon^{\alpha\beta}\bar{e}_\beta)^* \end{pmatrix}. \tag{3.8}$$

There are by now many references on supersymmetry, for an incomplete list see [1–5].

3.2 The MSSM

The generalization of the SM to the Minimal Supersymmetric Standard Model [MSSM] is now quite simple. One defines the left-handed chiral superfields. For example, the electron left-handed Weyl field, e, along with its scalar partner, $\tilde{e}$, is contained in the left-handed chiral superfield, E, with

$$E(y, \theta) = \tilde{e}(y) + \sqrt{2}(\theta\ e(y)) + (\theta\ \theta)F_e(y) \tag{3.9}$$

where $y^{\mu} = x^{\mu} - i\theta\ \sigma^{\mu}\theta^{*}$ and $F_e(y)$ is an auxiliary field. We shall also use the notation $\theta^2 \equiv (\theta\ \theta)$. Note, θ_α is a Grassmann coordinate, transforming as a left-handed Weyl spinor.

In addition, the gauge bosons are contained in a real superfield, $V(x, \theta, \theta^*)$, which, in the so-called Wess-Zumino gauge, has the form, $V_{WZ} = -\theta\ \sigma^{\mu}\ \theta^{*}\ V_{\mu}(x) + i\theta^2\ (\theta^*\ \lambda(x)^*) - i\theta^{*2}\ (\theta\ \lambda(x)) + \frac{1}{2}\theta^2\ \theta^{*2}\ D(x)$ where V_μ is the gauge field, λ_α is the gaugino and D is the auxiliary field. The superfield containing the gauge boson field strength is given by $\mathscr{W}^{\alpha}{}_{WZ} = g[\lambda^{\alpha}(y) + \theta^{\alpha}\ (-D(y)) + i\sigma^{\mu\nu\alpha\beta}\ \theta_\beta\ F_{\mu\nu}(y) + \theta^2\ (\sigma^{\mu}\ \partial_{\mu}\ \lambda(y)^*)^{\alpha}]$. From now on all gauge superfields will be given in the Wess-Zumino gauge.

Then the supersymmetric Lagrangian includes the gauge-kinetic term

$$\begin{aligned}\mathscr{L}_{gauge-kinetic} &= [\frac{1}{8g_s^2}\int d^2\theta Tr(\mathscr{W}_g{}^{\alpha}\ \mathscr{W}_{g\alpha}) + h.c.] \\ &+[\frac{1}{8g^2}\int d^2\theta Tr(\mathscr{W}_W{}^{\alpha}\ \mathscr{W}_{W\alpha}) + h.c.] \\ &+[\frac{1}{16g'^2}\int d^2\theta(\mathscr{W}_b{}^{\alpha}\ \mathscr{W}_{b\alpha}) + h.c.]\end{aligned} \tag{3.10}$$

and the gauge-matter interaction term

$$\mathscr{L}_{gauge-matter} = [\int d^4\theta \mathscr{K}] \tag{3.11}$$

where the Kähler potential, $\mathscr{K}$, is given by

$$\mathscr{K} = L_i^{\dagger}\ exp(-2\ [g_s V_g + g V_W + g'\frac{Y}{2}V_B])\ L_i + \cdots]. \tag{3.12}$$

Note, $V_g = V_A\ \mathscr{T}_A,\ V_W = V_a\ T_a$ and the flavor index $i = 1, 2, 3$.

The Yukawa interactions are then given by $\mathscr{L}_{Yukawa} = \int d^2\theta\mathscr{W} + h.c.$ with the superpotential given by

$$\begin{aligned}\mathscr{W} &= \epsilon_{\alpha\beta}\ \bar{E}_i\ Y_e^{ij}\ L_j^{\alpha}\ H_d^{\beta} + \epsilon_{\alpha\beta}\ \bar{D}_i\ Y_d^{ij}\ Q_j^{\alpha}\ H_d^{\beta} + \epsilon_{\alpha\beta}\ \bar{U}_i\ Y_u^{ij}\ Q_j^{\beta}\ H_u^{\alpha} \\ &+\epsilon_{\alpha\beta}\ \bar{N}_i\ Y_\nu^{ij}\ L_j^{\beta}\ H_u^{\alpha} - \frac{1}{2}M_{ij}\ \bar{N}_i\ \bar{N}_j - \mu\ \epsilon_{\alpha\beta}\ H_u^{\alpha}\ H_d^{\beta}.\end{aligned} \tag{3.13}$$

1. Note, in a supersymmetric generalization of the SM there are necessarily two Higgs doublets, $H_u,\ H_d$. This is because the superpotential is given in terms of the *analytic holomorphic product of superfields only* and H_u and H_d are both $SU(2)$ doublets, but with opposite hypercharge.

2. In addition, in order for the theory to be anomaly free, we need higgsinos with equal and opposite hypercharge to cancel the Y^3 and $Y\, SU(2)^2$ anomalies.
3. The theory has a discrete symmetry known as R-parity [6, 7] $= (-1)^{3(B-L)+2S}$ defined by the transformation

$$H(y,\theta) \to H(y,-\theta), \quad F(y,\theta) \to -F(y,-\theta) \tag{3.14}$$

   where H is either Higgs doublet and F represents any quark or lepton superfield. All ordinary SM particles have even charge under R-parity, while all superpartners have odd charge. As a consequence, superpartners must be produced in pairs in any accelerator experiment and the lightest supersymmetric particle [LSP] is absolutely stable. Thus the LSP is a good candidate for dark matter.
4. The MSSM is the most general Lagrangian extension of the SM consistent with $SU(3) \times SU(2) \times U(1)_Y$, supersymmetry and R-parity.
5. The component fields for sterile neutrinos are given by $\bar{N} \supset \{\tilde{\bar{\nu}},\ \bar{\nu},\ F_{\bar{\nu}}\}$.
6. The theory has a μ problem. The μ term is a supersymmetric contribution to the MSSM Lagrangian. The Higgs mass (and thus the Z mass) depends on the value of μ. But in the MSSM, there is no reason for μ to be of order the weak scale and not much, much larger.

3.3 MSSM in Terms of Component Fields

The Lagrangian for the MSSM written in terms of superfields is quite compact. However once the integrals over the Grassmann coordinates are done, the Lagrangian has many terms. In order to display them, let's discuss the form of the most general supersymmetric Lagrangian. Consider chiral superfields,

$$\Phi_i(y,\theta) = \phi_i(y) + \sqrt{2}(\theta\ \psi_i(y)) + \theta^2\ F_i(y) \tag{3.15}$$

and gauge superfield

$$\begin{aligned} V^s(x,\theta,\theta^*) &\equiv V^s(x,\theta,\theta^*)_a\ T^s_a \\ &= -\theta\ \sigma^\mu\ \theta^*\ V^s_\mu(x) + i\theta^2\ (\theta^*\ \lambda^s(x)^*) \\ &\quad -i\theta^{*2}\ (\theta\ \lambda^s(x)) + \frac{1}{2}\theta^2\ \theta^{*2}\ D^s(x) \end{aligned} \tag{3.16}$$

where $T^s_a,\ a = 1,\cdots,d^s_{adj}$ are the generators of the group $\mathcal{G}^s$ in the fundamental representation (or in the representation relevant for the chiral matter fields) and d^s_{adj} is the dimension of the adjoint representation. The supersymmetric gauge field strength is

$$\mathcal{W}^{s\alpha} = g^s[\lambda^s(y)^\alpha + \theta^\alpha\ (-D^s(y)) + i(\sigma^{\mu\nu})^{\alpha\beta}\ \theta_\beta\ F^s(y)_{\mu\nu} + \theta^2\ (\sigma^\mu\ \partial_\mu\ \lambda^s(y)^*)^\alpha]. \tag{3.17}$$

The Lagrangian has three terms

1.

$$\mathscr{L}_{gauge-kinetic} = \sum_s [\frac{1}{16\, k\, g^{s2}} \int d^2\theta Tr(f(\Phi_i)\, \mathscr{W}^{s\alpha}\, \mathscr{W}^s{}_\alpha) + h.c.] \tag{3.18}$$

2.

$$\mathscr{L}_{gauge-matter} = [\int d^4\theta \mathscr{K}] \tag{3.19}$$

where the renormalizable Kähler potential is given by

$$\mathscr{K} = \sum_i (\Phi_i^\dagger\; exp[\sum_s (-2\, g^s\, V^s)]\; \Phi_i). \tag{3.20}$$

In general, the Kähler potential is a hermitian, gauge invariant function of the fields, Φ_i, Φ_i^* and V^s. In addition, the chiral matter interactions are determined by the Yukawa Lagrangian,

3.

$$\mathscr{L}_{Yukawa} = \int d^2\theta \mathscr{W}(\Phi_i) + h.c \tag{3.21}$$

where the gauge kinetic function, $f(\Phi)$, and the superpotential, $\mathscr{W}(\Phi_i)$, are functions of products of the fields, Φ_i, and $k = 1$ for $U(1)$ gauge interactions and $k = \frac{1}{2}$ for $SU(N)$ interactions.

Upon integrating over the Grassmann coordinates we obtain the gauge kinetic Lagrangian (for $f(\Phi_i) = 1$ and renormalizable Kähler potential)[1]

1.

$$\mathscr{L}_{gauge-kinetic} = \sum_s \left[-\frac{1}{2} Tr(F^{s2}_{\mu\nu}) + 2iTr(\lambda^{s*} \bar{\sigma}^\mu\, D^s_\mu\, \lambda^s) + Tr(D^{s2}) \right]; \tag{3.22}$$

the gauge-matter interactions

2.

$$\mathscr{L}_{gauge-matter} = \sum_{i,s} \Big[i\psi_i^\dagger\, \bar{\sigma}^\mu\, D^s_\mu\, \psi_i + |D^s_\mu \phi_i|^2 + |F_i|^2 - (\sqrt{2} i g^s\, \phi_i^* (\lambda^s\, \psi_i) + h.c.) - g^s \phi_i^*\, D^s\, \phi_i \Big], \tag{3.23}$$

[1] For a $U(1)$ gauge theory divide each term in $\mathscr{L}_{gauge-kinetic}$ by a factor of 2.

and the superpotential terms

3.

$$\mathscr{L}_{Yukawa} = F_i \, \frac{\partial \mathscr{W}}{\partial \Phi_i}|_{\theta=0} - \frac{1}{2}\, \psi_i \, \psi_j \, \frac{\partial^2 \mathscr{W}}{\partial \Phi_i \, \partial \Phi_j}|_{\theta=0} + h.c.. \tag{3.24}$$

The equations of motion can be used to eliminate the terms containing the auxiliary fields, D^s and F_i. We then have $-F_i^* = \frac{\partial \mathscr{W}}{\partial \Phi_i}|_{\theta=0}$ and $(D^s)_a = g^s \, \sum_i (\phi_i^* \, T_a \, \phi_i)$. Plugging these solutions of the equations of motion back into the full Lagrangian, $\mathscr{L} = \mathscr{L}_{gauge-kinetic} + \mathscr{L}_{gauge-matter} + \mathscr{L}_{Yukawa}$ we obtain the scalar potential

$$V(\phi_i) = \sum_i |F_i|^2 + \frac{1}{2} \sum_a ((D^s)_a)^2. \tag{3.25}$$

Note, this Lagrangian is supersymmetric with equal masses for the Standard Model particles and their superpartners. This is clearly inconsistent with low energy data, since no superpartners have been discovered. Therefore it is necessary for supersymmetry to be broken. In addition, supersymmetry is an extension of Poincare invariance. And in the Standard Model, including gravity, Poincare invariance is a local symmetry, i.e. the consequence of local Poincare invariance is Einstein's theory of general relativity. Therefore supersymmetry must also be a local symmetry, resulting in supergravity. The superpartner of the spin 2 graviton is a spin 3/2 gravitino. Thus, if we need to break supersymmetry, to give mass to all the superpartners, we must do it spontaneously. As a consequence of spontaneously breaking supersymmetry, the gravitino obtains mass via a superHiggs mechanism. Then all the superpartners obtain mass, either at tree level or via radiative corrections.

3.4 MSSM Spectrum and Supersymmetric Interactions

The low energy spectrum of the MSSM includes the following:

- For every fermion (quark and lepton) in the SM, there is a supersymmetric scalar partner (squark ($\tilde{q}, \ \tilde{\bar{u}}, \ \tilde{\bar{d}}$) and slepton ($\tilde{l}, \ \tilde{\bar{e}}, \ \tilde{\bar{\nu}}$)) with both chiralities;
- For every gauge boson in the SM, there is a supersymmetric Weyl spinor partner (gluinos ($\tilde{g}$), winos ($\tilde{w}^{\pm}, \tilde{w}^0$), bino ($\tilde{b}$));
- Since there are two Higgs doublets in the MSSM, after H_u and H_d obtain VEVs $W^{\pm}$ and Z^0 obtain mass via the Higgs mechanism. There are then fine observable scalar Higgs states, $h, \ H$ (CP even Higgs scalars), A^0 (CP odd pseudo-scalar) and $H^{\pm}$ (charged Higgs). In addition for each Higgs doublet state there are supersymmetric fermionic partners (neutral ($\tilde{h}^0, \ \tilde{\bar{h}}^0$) and charged Higgsinos ($\tilde{h}^+, \ \tilde{\bar{h}}^-$).

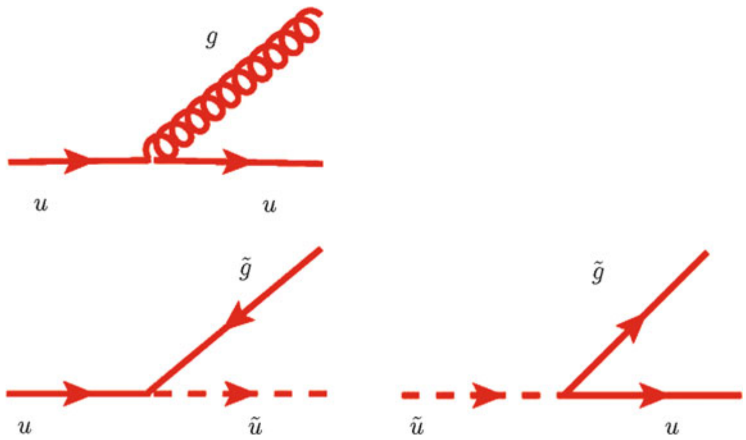

Fig. 3.1 SM gauge interaction above and new supersymmetric gaugino interactions below

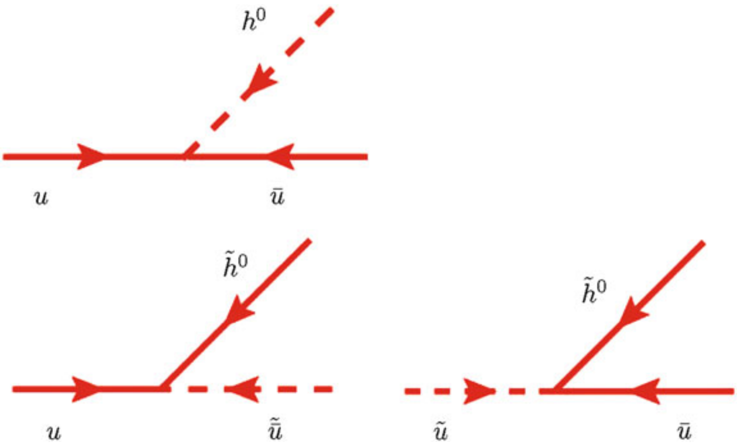

Fig. 3.2 SM Higgs interaction above and new supersymmetric Higgsino interactions below

Left and right-handed squarks and sleptons mix among states with the same SM charges. Thus there is a 6×6 up squark mass matrix and similarly for down squarks, charged and neutral sleptons. In addition the charged Higgsinos mix with the charged winos to form mass eigenstates, charginos ($\tilde{\chi}_i^{\pm}, \quad i = 1, 2$) and the neutral Higgsinos mix with the neutral wino and bino to form mass eigenstates, neutralinos ($\tilde{\chi}_i^0, \quad i = 1, 2, 3, 4$). The SM and new supersymmetric interactions of gauginos and Higgsinos are given in Figs. 3.1 and 3.2.

Chapter 4
Soft SUSY Breaking Mechanisms

In this lecture we consider the supersymmetry breaking mechanisms and their effects on low energy physics. We shall consider only those breaking mechanisms relevant for supersymmetry in 3+1 space-time dimensions. In later lectures we shall consider higher dimensional theories. In order to discuss spontaneous SUSY breaking, we must first discuss the actual symmetry. In the previous lecture we simply wrote down the most general supersymmetric Lagrangian without specifying the actual supersymmetry. In fact, the Lagrangian is invariant under global SUSY transformations

$$\Phi'(x^\mu,\ \theta_\alpha,\ \theta^*_\alpha) = \Phi(x^{\mu\prime},\ \theta'_\alpha,\ {\theta^*}'_\alpha) \tag{4.1}$$

with

$$\begin{aligned} x^{\mu\prime} &= x^\mu - i(\epsilon\ \sigma^\mu\ \theta^* - \theta\ \sigma^\mu\ \epsilon^*) \\ \theta'_\alpha &= \theta_\alpha + \epsilon_\alpha \\ {\theta^*}'_\alpha &= \theta^*_\alpha + \epsilon^*_\alpha \end{aligned} \tag{4.2}$$

where ϵ_α is a constant Grassmann parameter. Note, SUSY transformations are translations in superspace. If we expand the chiral superfield out to linear order in the parameter ϵ we find the transformation law on ordinary fields (with $\Phi(x, \theta_\alpha) = \{\phi(x),\ \psi_\alpha(x),\ F(x)\}$) given by

$$\begin{aligned} \delta\phi(x) &= \sqrt{2}(\epsilon\ \psi(x)) \\ \delta\psi_\alpha(x) &= \sqrt{2}\ F(x)\ \epsilon_\alpha - i\sqrt{2}\ (\sigma^\mu\ \epsilon^*)_\alpha\ \partial_\mu\phi(x) \\ \delta F(x) &= -\sqrt{2}i\ \epsilon^*\ \bar{\sigma}^\mu\ \partial_\mu\psi(x). \end{aligned} \tag{4.3}$$

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Similarly for the vector multiplet $W_\alpha(x, \theta) = \{F_{\mu\nu}(x),\ \lambda_\alpha(x), D(x)\}$ we have

$$\begin{aligned} \delta F_{\mu\nu}(x) &= i[\epsilon\ \sigma_\nu\ \partial_\mu\ \lambda^*(x) + \epsilon^*\ \bar{\sigma}_\nu\ \partial_\mu \lambda(x) - (\mu \leftrightarrow \nu)] \\ \delta\lambda_\alpha(x) &= i\epsilon_\alpha\ D(x) + (\sigma^{\mu\nu}\epsilon)_\alpha\ F_{\mu\nu}(x) \\ \delta D(x) &= \epsilon^*\ \bar{\sigma}^\mu\ \partial_\mu \lambda(x) - \epsilon\ \sigma^\mu \partial_\mu \lambda^*(x). \end{aligned} \tag{4.4}$$

Note, for both the chiral and vector supermultiplets the auxiliary fields, $F(x),\ D(x)$, transform as total derivatives. In fact, the F term of any product of chiral superfields also transforms as a total derivative. This is why the superpotential term and the gauge kinetic term are, by themselves, supersymmetric since they are given as an F term of a product of chiral superfields. In addition, any D term of a real superfield transforms as a total derivative. This is why the Kähler term is also supersymmetric.

We also now see how to spontaneously break supersymmetry. If the vacuum expectation value of either $F(x)$ or $D(x)$ are non vanishing space time constants, then supersymmetry is spontaneously broken. In fact, we can see by taking the vacuum expectation value of both sides of the transformation (where Q_α is the Weyl spinor supercharge)

$$i[(\epsilon Q),\ \psi_\alpha] = \sqrt{2}\, F(x)\, \epsilon_\alpha \tag{4.5}$$

and

$$i[(\epsilon Q),\ \lambda_\alpha] = i\epsilon_\alpha\ D(x), \tag{4.6}$$

that SUSY is spontaneously broken when $\langle F(x) \rangle = F$ or $\langle D(x) \rangle = D$. Note, as a result of spontaneous symmetry breaking, the supercurrent creates a massless Goldstone fermion, called the goldstino, out the vacuum. Moreover, the energy in the vacuum is non-zero [see Eq. (3.25)]. This is true for global supersymmetry. When one includes gravity, with local SUSY, then a zero vacuum energy can be obtained; albeit by fine-tuning.

4.1 SUSY Non-renormalization Theorems

Supersymmetric field theories have special properties which set them apart from all other field theories. They possess what are known as *non-renormalization theorems* which have the following consequences.[1]

[1]See problem 2.

1. There are no vertex corrections (even finite) to the superpotential, $\mathscr{W}$ [8]. As a consequence, renormalization of parameters in the theory are only due to wave function renormalization.
2. In $U(1)$ gauge theories, there may be one loop corrections to the Fayet-Illiopoulos D-term. These corrections are of the form $\varDelta D \sim g^2 \sum_i q_i^2 \varLambda^2$ where g is the gauge coupling constant, q_i are the $U(1)$ charges for the chiral superfields and $\varLambda$ is the cut-off. These corrections would produce quadratic radiative corrections to scalar masses. However, it has been proven [9] that the radiative corrections to the D-term only exist at one loop. Thus if the one loop correction vanishes, the corrections to the D-term vanish to all orders in perturbation theory. In particular, in the MSSM there is no one loop correction to the $U(1)_Y$ D-term, since the sum over the hypercharge of quarks and leptons vanishes.
3. If SUSY is unbroken at tree level, then it is unbroken to all finite orders in perturbation theory.
4. There are no quadratic radiative corrections to scalar masses.[2] This has been discussed in the context of solving the gauge hierarchy problem (why is the weak scale, M_W, so much smaller than the Planck scale,M_{Pl}) [10, 11] by Witten [12, 13].
5. In the global supersymmetric limit the vacuum energy vanishes, since

$$V(\phi_i) = \sum_i |F_i|^2 + \frac{1}{2} \sum_a ((D^s)_a)^2. \tag{4.7}$$

In a non-renormalizable, SUSY Lagrangian both the Kähler potential and the superpotential can contain higher dimension operators and the gauge kinetic function can be non-trivial. In this case, the following notation becomes convenient. We define derivatives with respect to $\mathscr{K}$ by

$$\mathscr{K}_i = \frac{\partial \mathscr{K}}{\partial \varPhi_i}|_{\theta=0}, \quad g_{i\bar{j}} \equiv \mathscr{K}_{i\bar{j}} = \frac{\partial^2 \mathscr{K}}{\partial \varPhi_i \, \partial \bar{\varPhi}_j}|_{\theta=0} \tag{4.8}$$

where $g_{i\bar{j}} \equiv \mathscr{K}_{i\bar{j}}$ is the Kähler metric. Then the matter kinetic terms are no longer canonical and the equation for the auxiliary fields become

$$- F_i^* = g_{\bar{i}j} \frac{\partial \mathscr{W}}{\partial \varPhi_j}|_{\theta=0} \tag{4.9}$$

and

$$(D^s)_a = -(Ref(\phi)^{-1})_{ab} g^s \sum_i (\mathscr{K}_i \, T_b \, \phi_i). \tag{4.10}$$

[2] This assumes the one loop correction to the $U(1)$ D-term vanishes.

The vacuum energy is now given by

$$V(\phi_i) = \sum_{i\bar{j}} g^{i\bar{j}} \frac{\partial \mathscr{W}}{\partial \Phi_i} (\frac{\partial \mathscr{W}}{\partial \Phi_j})^* + \frac{1}{2} \sum_{ab} (Ref(\phi))_{ab} (D^s)_a (D^s)_b. \tag{4.11}$$

Local supersymmetry is supergravity. In the case of local supersymmetry, the auxiliary fields become more complicated. We define the Kähler covariant derivative, $D_i = \frac{\partial}{\partial \phi_i} + \frac{1}{m_{Pl}^2} \mathscr{K}_i$ where m_{Pl} is the reduced Planck scale, i.e. $m_{Pl} = M_{Pl}/\sqrt{8\pi}$. Then $F^i = -e^{\mathscr{K}/2m_{Pl}^2} g^{i\bar{j}} D_{\bar{j}} \mathscr{W}^*$ and the scalar potential becomes

$$e^{\mathscr{K}/m_{Pl}^2} [D_i \mathscr{W} g^{i\bar{j}} D_{\bar{j}} \mathscr{W}^* - 3\frac{1}{m_{Pl}^2} |\mathscr{W}|^2] + \frac{1}{2} \sum_{ab} (Ref(\phi))_{ab} (D^s)_a (D^s)_b. \tag{4.12}$$

If $D_i \mathscr{W} = 0$ and $D_a = 0$, then SUSY is unbroken.[3] On the other hand, if for example $D_i \mathscr{W} \neq 0$ then SUSY is broken spontaneously. As a consequence there is a superHiggs mechanism. The goldstino is eaten by the gravitino and the spin 3/2 gravitino becomes massive with $m_{3/2} = e^{\mathscr{K}/2m_{Pl}^2} \frac{\mathscr{W}}{m_{Pl}^2}$ when $V(\langle \phi \rangle) = 0$.

Historically, the earliest breaking mechanism discussed (in the context of global SUSY) was either the O'Raifeartaigh mechanism [14] or the Fayet-Iliopoulos [FI] D term mechanism [15] (see the review article by Fayet and Ferrara [16]).[4] In both cases SUSY breaking effects were transmitted to the Standard Model sector directly. Without specifying the mechanisms here, this caused two problems. In the case of the O'Raifeartaigh mechanism it can be shown that the sum over all mass squareds, i.e.

$$\sum_J (-1)^{2J} (2J+1) m_J^2 = 0. \tag{4.13}$$

As a consequence it is easy to see that the lightest SUSY partner of quarks and leptons was a scalar quark or lepton. Gauginos only obtain mass at the loop level. In the latter case, with a FI D term, it required the addition of an anomalous $U(1)$ gauge interaction which is unacceptable.

It was clear, early on, that transmitting SUSY breaking to the SM sector at tree level was a problem. It was also clear that transmitting SUSY breaking to the SM sector via loop corrections would solve these problems. Firstly, the supertrace formula, Eq. (4.13), was only a tree level problem. But in addition, since scalars suffer from a hierarchy problem which is solved by SUSY, then when SUSY is broken radiative corrections to scalar masses are expected to be of order $\frac{\alpha}{4\pi} \Lambda_{eff}^2$ where α is some coupling constant and Λ_{eff} is the effective SUSY breaking

[3] Note, if SUSY is unbroken at the tree level and $\langle \mathscr{W} \rangle = 0$, then the non-renormalization theorems guarantee that it is still unbroken to any finite order in perturbation theory.

[4] See problem 3.

scale. Moreover, gauginos only get radiative corrections to their mass when two symmetries are broken, supersymmetry and chiral symmetry. Thus one might expect gauginos to be lighter than scalars.

The two mechanisms for transmitting SUSY breaking in four space-time dimensions are *gauge-mediated* SUSY breaking [17–27] and *gravity mediated* SUSY breaking (with a special case given by gaugino condensation [28]).[5] In all cases, SUSY is typically broken in a hidden sector by a perturbative or non-perturbative O'Raifeartaigh and/or Fayet-Iliopoulos type mechanism and then transmitted to the visible sector by messenger fields through radiative corrections due to the SM gauge interactions or directly via gravity.[6]

The consequence of spontaneous SUSY breaking in the effective low energy theory, i.e. at energies much below the messenger scale, is soft SUSY breaking corrections to the MSSM Lagrangian. These soft SUSY breaking terms were first considered in [29] and studied in great detail by Girardello and Grisaru [30] who showed that the soft breaking terms preserve the property of no quadratic corrections to scalar masses.

4.2 Soft SUSY Breaking Lagrangian

The soft SUSY breaking terms considered in the MSSM are as follows:

$$\begin{aligned}
-\mathscr{L}_{soft} = {} & \tilde{q}^\dagger\, m_q^2\, \tilde{q} + \tilde{\bar{u}}^\dagger\, m_{\bar{u}}^2\, \tilde{\bar{u}} + \tilde{\bar{d}}^\dagger\, m_{\bar{d}}^2\, \tilde{\bar{d}} + \tilde{l}^\dagger\, m_l^2\, \tilde{l} + \tilde{\bar{e}}^\dagger\, m_{\bar{e}}^2\, \tilde{\bar{e}} + \tilde{\bar{\nu}}^\dagger\, m_{\bar{n}}^2\, \tilde{\bar{\nu}} \\
& + m_{h_u}^2\, H_u^\dagger\, H_u + m_{h_d}^2\, H_d^\dagger\, H_d \\
& + \epsilon_{\alpha\beta}\, \tilde{\bar{e}}_i\, A_e^{ij}\, \tilde{l}_j^\alpha\, H_d^\beta + \epsilon_{\alpha\beta}\, \tilde{\bar{d}}_i\, A_d^{ij}\, \tilde{q}_j^\alpha\, H_d^\beta + \epsilon_{\alpha\beta}\, \tilde{\bar{u}}_i\, A_u^{ij}\, \tilde{q}_j^\beta\, H_u^\alpha \\
& + \epsilon_{\alpha\beta}\, \tilde{\bar{\nu}}_i\, A_\nu^{ij}\, \tilde{l}_j^\beta\, H_u^\alpha - \frac{1}{2} B_\nu\, M_{ij}\, \tilde{\bar{\nu}}_i\, \tilde{\bar{\nu}}_j - B\, \mu\, \epsilon_{\alpha\beta}\, H_u^\alpha\, H_d^\beta \\
& - \frac{1}{2} M_3\, \lambda^A\, \lambda^A - \frac{1}{2} M_2\, \lambda^a\, \lambda^a - \frac{1}{2} M_1\, \lambda\, \lambda .
\end{aligned} \tag{4.15}$$

The scalar masses squared matrices are hermitian, the A parameters are, in general, arbitrary 3×3 complex matrices with dimensions of mass and B_ν, B

[5] Anomaly mediated SUSY breaking will be considered in the context of orbifold SUSY GUTs in Sect. 17.

[6] In supergravity Eq. (4.13) receives an important correction, i.e. for a flat Kähler potential it becomes

$$\sum_J (-1)^{2J} (2J+1) m_J^2 = 4 m_{3/2}^2. \tag{4.14}$$

and the gaugino masses $M_i, i = 1, 2, 3$ all have dimensions of mass. In addition all dimensionful parameters are assumed to be of order the weak scale. We will be more precise regarding the soft SUSY breaking parameters when we discuss phenomenology (see Chap. 7).

For now we define the so-called constrained MSSM [CMSSM] parameters. These are taken to be values defined at a GUT scale. We have universal

- scalar mass:

$$m_q^2|_{M_G} = m_u^2|_{M_G} = m_d^2|_{M_G} = m_l^2|_{M_G} = m_e^2|_{M_G} = m_n^2|_{M_G} \equiv m_0^2 \, 1_{3\times 3}; \tag{4.16}$$

- A parameter:

$$A_u^{ij} = A_0 \, Y_u^{ij}, \;\; A_d^{ij} = A_0 \, Y_d^{ij}, \;\; A_e^{ij} = A_0 \, Y_e^{ij}, \;\; A_\nu^{ij} \equiv A_0 \, Y_\nu^{ij}, \tag{4.17}$$

  and
- gaugino mass: $M_1 = M_2 = M_3 \equiv M_{1/2}$.

In the CMSSM, the Higgs mass parameters are universal and equal to m_0. However there are simple generalizations of the CMSSM with non-universal Higgs masses [NUHM].

- In NUHM1 we have $m_{H_u}^2 = m_{H_d}^2 \neq m_0^2$ and
- in NUHM2 we have $m_{H_u}^2 \neq m_{H_d}^2$ are both independent of m_0^2.

Finally, in any MSSM there is necessarily a gravitino mass parameter, $m_{3/2}$.

In the next lecture, we will review grand unified theories [GUTs]. There are also several good references for GUTs, see for example [31, 32].

Chapter 5
Introduction to $SU(5)$ and $SO(10)$ GUTs

5.1 Two Roads to Grand Unification

One can first unify quarks and leptons into two irreducible representations of the group $SU(4)_C \otimes SU(2)_L \otimes SU(2)_R$, i.e. the so-called Pati-Salam group [33–35] where lepton number is the fourth color.

Then the PS fields

$$\mathcal{Q} = (q\ l),\ \bar{\mathcal{Q}} = (\bar{q}\ \bar{l}) \tag{5.1}$$

where

$$\bar{q} = \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix},\ \bar{l} = \begin{pmatrix} \bar{\nu} \\ \bar{e} \end{pmatrix} \tag{5.2}$$

transform as $(4, 2, 1) \oplus (\bar{4}, 1, \bar{2})$ under PS. One can check that baryon number minus lepton number acting on a 4 of $SU(4)$ is given by

$$B - L = \begin{pmatrix} \frac{1}{3} & & & \\ & \frac{1}{3} & & \\ & & \frac{1}{3} & \\ & & & -1 \end{pmatrix} \tag{5.3}$$

and similarly electric charge is given by

$$Q = T_{3L} + T_{3R} + \frac{1}{2}(B - L). \tag{5.4}$$

S. Raby, *Supersymmetric Grand Unified Theories*, Lecture Notes in Physics 939,
DOI 10.1007/978-3-319-55255-2_5

Note, charge is quantized since it is embedded in a non-abelian gauge group. One family is contained in two irreducible representations. Finally, if we require parity ($L \leftrightarrow R$) then there are two independent gauge couplings.

What about the Higgs? The two Higgs doublets $H_u,\ H_d$ are combined into one irreducible PS Higgs multiplet (represented as a 2×2 matrix)

$$\mathscr{H} = (H_d,\ H_u) \tag{5.5}$$

transforming as a $(1, 2, \bar{2})$ under PS. Thus for one family, there is a unique renormalizable Yukawa coupling given by

$$\lambda\ \bar{\mathscr{Q}}\ \mathscr{H}\ \mathscr{Q} \tag{5.6}$$

giving the GUT relation

$$\lambda_t = \lambda_b = \lambda_\tau = \lambda_{\nu_\tau} \equiv \lambda\,. \tag{5.7}$$

Now Pati-Salam is *not* a grand unified gauge group. However, since $SU(4) \approx SO(6)$ and $SU(2) \otimes SU(2) \approx SO(4)$ (where $\approx$ signifies a homomorphism), it is easy to see that PS $\approx SO(6) \otimes SO(4) \subset SO(10)$ [36, 37]. In fact one family of quarks and leptons is contained in the spinor representation of $SO(10)$ (for more details, see Sect. 5.7), i.e.

$$\begin{aligned} &SO(10) \to SU(4)_C \otimes SU(2)_L \otimes SU(2)_R \\ &16 \to (4, 2, 1) \oplus (\bar{4}, 1, \bar{2}). \end{aligned} \tag{5.8}$$

Hence by going to $SO(10)$ we have obtained quark-lepton unification (one family contained in one spinor representation) and gauge coupling unification (one gauge group) (see Table 5.1).

But I should mention that there are several possible breaking patterns for $SO(10)$.

$$\begin{aligned} SO(10) &\to SU(4)_C \otimes SU(2)_L \otimes SU(2)_R \\ &\to SU(5) \otimes U(1)_X \\ &\to SU(5)' \otimes U(1)_{X'} \\ &\to SU(3)_C \otimes U(1)_{(B-L)} \otimes SU(2)_L \otimes SU(2)_R. \end{aligned} \tag{5.9}$$

In order to preserve a prediction for gauge couplings we would require the breaking pattern

$$SO(10) \to SM \tag{5.10}$$

or

$$SO(10) \to SU(5) \to SM. \tag{5.11}$$

Table 5.1 Spinor representation of $SO(10)$ where this table explicitly represents the Cartan-Weyl weights for the states of one family of quarks and leptons

Grand Unification − SO(10)			
State	Y $= -\frac{2}{3}\Sigma(\mathrm{C}) + \Sigma(\mathrm{W})$	Color C spins	Weak W spins
$\bar{\boldsymbol{\nu}}$	0	$--,-$	$--$
$\bar{\mathbf{e}}$	2	$---$	$++$
$\mathbf{u_r}$	1/3	$+--$	$-+$
$\mathbf{d_r}$	1/3	$+--$	$+-$
$\mathbf{u_b}$	1/3	$-+-$	$-+$
$\mathbf{d_b}$	1/3	$-+-$	$+-$
$\mathbf{u_y}$	1/3	$--+$	$-+$
$\mathbf{d_y}$	1/3	$--+$	$+-$
$\bar{\mathbf{u}}_\mathbf{r}$	−4/3	$-++$	$--$
$\bar{\mathbf{u}}_\mathbf{b}$	−4/3	$+-+$	$--$
$\bar{\mathbf{u}}_\mathbf{y}$	−4/3	$++-$	$--$
$\bar{\mathbf{d}}_\mathbf{r}$	2/3	$-++$	$++$
$\bar{\mathbf{d}}_\mathbf{b}$	2/3	$+-+$	$++$
$\bar{\mathbf{d}}_\mathbf{y}$	2/3	$++-$	$++$
$\boldsymbol{\nu}$	−1	$+++$	$-+$
$\mathbf{e}$	−1	$+++$	$+-$

The double lines separate irreducible representations of $SU(5)$

5.2 Introduction to $SU(5)$

It will be convenient at times to work with the Georgi-Glashow GUT group $SU(5)$ [38, 39]. We have $16 \to 10 \oplus \bar{5} \oplus 1$. Let's identify the quarks and leptons of one family directly. We define the group $SU(5)$ by

$$SU(5) = \{U | U = 5 \times 5 \text{ complex matrix}; U^\dagger U = 1; detU = 1\} \tag{5.12}$$

and the fundamental representation $5^\alpha,\ \alpha = 1, \ldots, 5$ transforms as

$$5'^\alpha = U^\alpha{}_\beta\, 5^\beta. \tag{5.13}$$

We represent the unitary matrix U by

$$U = \exp(iT_A\, \omega_A) \tag{5.14}$$

where $Tr(T_A) = 0,\ T_A^\dagger = T_A, A = 1, \ldots, 24$ and $[T_A,\ T_B] = if_{ABC}\, T_C$ with f_{ABC} the structure constants of $SU(5)$. Under an infinitesimal transformation, we have

$$\delta_A 5^\alpha = i(T_A)^\alpha{}_\beta\, 5^\beta\, \omega_A. \tag{5.15}$$

Let us now identify the $SU(3) \otimes SU(2) \otimes U(1)_Y$ subgroup of $SU(5)$. The $SU(3)$ subgroup is given by the generators

$$T_A = \left(\begin{array}{c|c} \frac{1}{2}\lambda_A & 0 \\ \hline 0 & 0 \end{array}\right), \quad A = 1, \ldots, 8. \tag{5.16}$$

And the $SU(2)$ subgroup is given by

$$T_A = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \frac{1}{2}\tau_{(A-20)} \end{array}\right), \quad A = 21, 22, 23. \tag{5.17}$$

The generators in $SU(5)/SU(3) \otimes SU(2) \otimes U(1)_Y$ are given by

$$T_A, \; A = 9, \ldots, 20. \tag{5.18}$$

These are 12 generators of the form

$$\frac{1}{2}\left(\begin{array}{c|c} & 1\;0 \\ 0 & 0\;0 \\ & 0\;0 \\ \hline 1\;0\;0 & 0 \\ 0\;0\;0 & \end{array}\right), \quad \frac{1}{2}\left(\begin{array}{c|c} & -i\;0 \\ 0 & 0\;\;0 \\ & 0\;\;0 \\ \hline i\;0\;0 & 0 \\ 0\;0\;0 & \end{array}\right). \tag{5.19}$$

Let us now identify the hypercharge Y. The only remaining generator of $SU(5)$ commuting with the generators of $SU(3)$ and $SU(2)$ is given by

$$T_{24} = \sqrt{\frac{3}{5}}\left(\begin{array}{ccc|cc} -1/3 & 0 & 0 & & \\ 0 & -1/3 & 0 & 0 & \\ 0 & 0 & -1/3 & & \\ \hline & 0 & & 1/2 & 0 \\ & & & 0 & 1/2 \end{array}\right) \equiv \sqrt{\frac{3}{5}}\,\frac{Y}{2}. \tag{5.20}$$

The overall normalization is chosen so that all the $SU(5)$ generators satisfy $Tr(T_A\, T_B) = \frac{1}{2}\,\delta_{AB}$.

With these identifications, we see that the quantum numbers of a

$$5^\alpha = \begin{pmatrix} d^a \\ \bar{l}^i \end{pmatrix}, \; a = 1, 2, 3; \quad i = 4, 5 \tag{5.21}$$

where d^a transforms as $(3, 1, -2/3)$ and l^i transforms as $(1, 2, +1)$ under the SM. Of course these are not correct quantum numbers for any of the quarks and leptons, but the charge conjugate states are just right. We have

$$\bar{5}_\alpha = \begin{pmatrix} \bar{d}_a \\ l'_i \end{pmatrix}, \; a = 1, 2, 3; \quad i = 4, 5 \tag{5.22}$$

with transformation properties

$$\bar{d} = (\bar{3}, 1, 2/3), \quad l' = (1, \bar{2}, -1) \tag{5.23}$$

and

$$l' = \begin{pmatrix} -e \\ \nu \end{pmatrix}. \tag{5.24}$$

Once we have identified the states of the $\bar{\mathbf{5}}$, we have no more freedom for the **10**.[1] The **10** transforms as an anti-symmetric tensor product of two **5**s, i.e.

$$10^{\alpha\beta} = -10^{\beta\alpha} \propto 5_1^{\alpha}\, 5_2^{\beta} - 5_2^{\alpha}\, 5_1^{\beta}. \tag{5.25}$$

We find

$$\begin{aligned} 10^{ab} &\equiv \epsilon^{abc}(\bar{u})_c = (\bar{3}, 1, -4/3) \\ 10^{ai} &\equiv q^{ai} = (3, 2, 1/3) \\ 10^{ij} &\equiv \epsilon^{ij}\bar{e} = (1, 1, +2). \end{aligned} \tag{5.26}$$

To summarize we find

$$\bar{5}_\alpha = \begin{pmatrix} \bar{d}_1 \\ \bar{d}_2 \\ \bar{d}_3 \\ -e \\ \nu \end{pmatrix}, \quad 10^{\alpha\beta} = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|cc} 0 & \bar{u}_3 & -\bar{u}_2 & u^1 & d^1 \\ -\bar{u}_3 & 0 & \bar{u}_1 & u^2 & d^2 \\ \bar{u}_2 & -\bar{u}_1 & 0 & u^3 & d^3 \\ \hline -u^1 & -u^2 & -u^3 & 0 & \bar{e} \\ -d^1 & -d^2 & -d^3 & -\bar{e} & 0 \end{array}\right). \tag{5.27}$$

What about the Higgs doublets. Clearly since the smallest representation of $SU(5)$ is five dimensional we need to extend the Higgs sector and add color triplet Higgs bosons. We define

$$H = \begin{pmatrix} T \\ H_u \end{pmatrix}, \quad \bar{H} = \begin{pmatrix} \bar{T} \\ H'_d \end{pmatrix} \tag{5.28}$$

transforming as a **5** and $\bar{\mathbf{5}}$ and $(H'_d)_\alpha \equiv \epsilon_{\alpha\beta}\, H_d^\beta = \begin{pmatrix} -\bar{h}^- \\ \bar{h}^0 \end{pmatrix}$.

[1] See problem 4.

Now that we have identified the states of one family in $SU(5)$, let us exhibit the fermion Lagrangian (with gauge interactions). We have

$$\mathscr{L}_{fermion} = \bar{5}^{\dagger}_{\alpha} i(\bar{\sigma}_{\mu}\, D^{\mu})_{\alpha}{}^{\beta}\, \bar{5}_{\beta} + \frac{1}{2}\, 10^{\alpha\beta\dagger} i(\bar{\sigma}_{\mu}\, D^{\mu})^{\alpha\beta}_{\gamma\delta}\, 10^{\gamma\delta} \tag{5.29}$$

where

$$D^{\mu} = \partial^{\mu} + i g_5 T_A\, A^{\mu}_A \tag{5.30}$$

and T_A is in the $\bar{\mathbf{5}}$ or $\mathbf{10}$ representation. We see that since there is only one gauge coupling constant at the GUT scale we have

$$g_3 = g_2 = g_1 \equiv g_5 \tag{5.31}$$

where, after weak scale threshold corrections are included, we have

$$g_3 \to g_s,\; g_2 \to g,\; g_1 \to \sqrt{\frac{5}{3}}\, g'. \tag{5.32}$$

At the GUT scale we have the relation

$$\sin^2\theta_W = \frac{(g')^2}{g^2 + (g')^2} = 3/8. \tag{5.33}$$

This result then gets renormalized to the weak scale, where it is measured.

5.3 Spontaneous Breaking of $SU(5)$ to the SM

We have assumed that $SU(5)$ is spontaneously broken via a Higgs mechanism to the Standard Model. The simplest way to accomplish this is to add a new Higgs multiplet in the adjoint representation of $SU(5)$. Consider adding the real scalar fields, Σ_A, and $\Sigma = \sum_{A=1}^{24} \Sigma_A\, T_A$. Then construct a scalar potential such that $\langle\Sigma\rangle \sim M_G\, T_{24}$. Since, under an $SU(5)$ transformation U, we have $\Sigma' = U\,\Sigma\,U^{\dagger}$, it is easy to see that the unbroken gauge symmetry is $SU(3)\times SU(2)\times U(1)_Y$. The only other requirement is that the only massless sector after symmetry breaking is the Standard Model matter sector. Thus one also needs to couple Σ to the Higgs, $\mathbf{5}$ and $\bar{\mathbf{5}}$ in such a way that only the Higgs doublets are light, while the color triplet Higgs have mass greater than of order 10^{10} GeV [40–42]. This is because the color triplet Higgs scalars also mediate proton decay. However, they only couple to quarks and leptons proportional to small Yukawa couplings. For example the Σ Lagrangian may be of the form

$$\begin{aligned}\mathscr{L}_{\Sigma} &= Tr(D_{\mu}\,\Sigma)^2 - Tr(\Sigma^2 - M_G^2)^2 \\ &\quad -(\lambda\, H^{\dagger}\,\Sigma\, H + \bar{\lambda}\,\bar{H}\,\Sigma\,\bar{H}^{\dagger}) - m_H^2\, H^{\dagger} H - m_{\bar{H}}^2\,\bar{H}^{\dagger}\,\bar{H}\end{aligned} \tag{5.34}$$

where the terms containing the Higgs fields have to be fine tuned to obtain light Higgs doublets and heavy color triplet Higgses.[2]

5.4 Gauge Coupling Unification Without SUSY

The tree level relations, Eq. (5.31), do not take into account threshold corrections at either the GUT or the weak scales or renormalization group [RG] running from the GUT scale to the weak scale [39, 43–46]. Consider first RG running. The one loop RG equations are given by

$$\frac{d\alpha_i}{dt} = -\frac{b_i}{2\pi}\alpha_i^2 \tag{5.35}$$

where $\alpha_i = \frac{g_i^2}{4\pi}$, $i = 1, 2, 3$ and

$$b_i = \frac{11}{3}C_2(G_i) - \frac{2}{3}T_R\, N_F - \frac{1}{3}T_R\, N_S. \tag{5.36}$$

Note, $t = -\ln(\frac{M_G}{\mu})$, and given

$$\sum_A (T_A^2) = C_2(G_i)\mathbb{1} \tag{5.37}$$

with T_A in the adjoint representation, this defines the quadratic Casimir for the group G_i with $C_2(SU(N)) = N$ and $C_2(U(1)) = 0$.

$$Tr(T_A T_B) = T_R\, \delta_{AB} \tag{5.38}$$

for T_A in the representation R (for $U(1)_Y$, $T_R \equiv \frac{3}{5}\, Tr(\frac{Y^2}{4})$) and $N_F(N_S)$ is the number of Weyl fermions (complex scalars) in representation R. The solution to the one loop RG equations is given by

$$\alpha_i(M_Z)^{-1} = \alpha_G^{-1} - \frac{b_i}{2\pi}\ln(\frac{M_G}{M_Z}). \tag{5.39}$$

For the SM we find

$$\mathbf{b}_{SM} \equiv (b_1, b_2, b_3) = (-\frac{4}{3}N_{fam} - \frac{1}{10}N_H, \frac{22}{3} - \frac{4}{3}N_{fam} - \frac{1}{6}N_H, 11 - \frac{4}{3}N_{fam}) \tag{5.40}$$

where $N_{fam}(N_H)$ is the number of families (Higgs doublets).

[2]See problem 5.

The one loop equations can be solved for the value of the GUT scale M_G and α_G in terms of the values of $\alpha_{EM}(M_Z)$ and $\sin^2\theta_W(M_Z)$. We have (without including weak scale threshold corrections)

$$\alpha_2(M_Z) = \frac{\alpha_{EM}(M_Z)}{\sin^2\theta_W(M_Z)}, \quad \alpha_1(M_Z) = \frac{\frac{5}{3}\alpha_{EM}(M_Z)}{\cos^2\theta_W(M_Z)} \tag{5.41}$$

and we find

$$\left(\frac{3}{5} - \frac{8}{5}\,\sin^2\theta_W(M_Z)\right)\alpha_{EM}(M_Z)^{-1} = (\frac{b_2 - b_1}{2\pi})\ln(\frac{M_G}{M_Z}) \tag{5.42}$$

which we use to solve for M_G. Then we use

$$\alpha_G^{-1} = \sin^2\theta_W(M_Z)\,\alpha_{EM}(M_Z)^{-1} + \frac{b_2}{2\pi}\ln(\frac{M_G}{M_Z}) \tag{5.43}$$

to solve for α_G. We can then predict the value for the strong coupling using

$$\alpha_3(M_Z)^{-1} = \alpha_G^{-1} - \frac{b_3}{2\pi}\ln(\frac{M_G}{M_Z}). \tag{5.44}$$

See Fig. 5.1 for the one loop running of the gauge couplings in the Standard Model and in the Minimal Supersymmetric Standard Model.

Given the experimental values $\sin^2\theta_W(M_Z) \approx 0.23$ and $\alpha_{EM}(M_Z)^{-1} \approx 128$ we find $M_G \approx 1.3\times10^{13}$ GeV with $N_H = 1$ and $\alpha_G^{-1} \approx 42$ for the SM with the one loop prediction for $\alpha_3(M_Z) \approx 0.07$. How well does this agree with the data? According

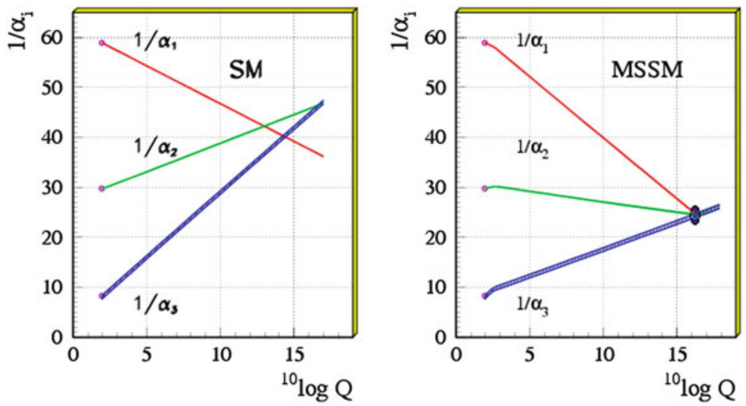

Fig. 5.1 One loop running of the gauge couplings in the Standard Model and in the MSSM

to the PDG the average value of $\alpha_s(M_Z) = 0.1185 \pm 0.0006$ [47]. So at one loop non-SUSY GUTs are clearly excluded.[3]

5.5 Fermion Mass Relations in $SU(5)$

Up and down quark Yukawa couplings at M_{GUT} are given in terms of the operators

$$-\mathscr{L}_{Yukawa} \supset \frac{1}{4}\,(Y_u)^{ij}\;10_i^{\alpha\beta}\;10_j^{\gamma\delta}\;H^{\eta}\;\epsilon_{\alpha\beta\gamma\delta\eta} + (Y_d)^{ij}\;\bar{H}_\alpha\;10_i^{\alpha\beta}\;\bar{5}_{j\beta}. \tag{5.45}$$

When written in terms of quark and lepton states we obtain the Yukawa couplings to the Higgs doublets

$$-\mathscr{L}_{Yukawa} \supset (Y_u)^{ij}\;\bar{u}_i\;q_j\;H_u + (Y_d)^{ij}\;(\bar{d}_j\;q_i + \bar{e}_i\;l_j')\;H_d'. \tag{5.46}$$

We see that $SU(5)$ relates the Yukawa couplings of down quarks and charged leptons, i.e. $\lambda_d = \lambda_e$ at the GUT scale. *Assuming this relation holds for all three families*, we have [49, 50] (for SUSY $SU(5)$ see [51–54]) $\lambda_b = \lambda_\tau, \quad \lambda_s = \lambda_\mu, \quad \lambda_d = \lambda_e$ at M_{GUT}.

To compare with experiment we must use the renormalization group[RG] equations to run these relations (valid at M_{GUT}) to the weak scale. The first relation gives a prediction for the b-τ ratio which is in good agreement with low energy data. Note, for heavy top quarks we must now use the analysis which includes the third generation Yukawa couplings [52]. We will discuss these results shortly. The next two relations can be used to derive the relation: $\frac{\lambda_s}{\lambda_d} = \frac{\lambda_\mu}{\lambda_e}$ at M_{GUT}. However at one loop the two ratios are to a good approximation RG invariants. Thus the relation is valid at any scale $\mu < M_{GUT}$. This leads to the *bad* prediction

$$\frac{m_s}{m_d} = \frac{m_\mu}{m_e}$$

for running masses evaluated at 2 GeV. It is a bad prediction since experimentally the left hand side is $\sim$20 while the right hand side is $\sim$200.

An ingenious method to fix this bad relation was proposed by Georgi and Jarlskog [55]. They show how to use $SU(5)$ Clebschs in a novel texture for fermion Yukawa matrices to keep the good relation $\lambda_b = \lambda_\tau$, and replace the bad relation above by the good relation

$$\frac{m_s}{m_d} = \frac{1}{9}\frac{m_\mu}{m_e}. \tag{5.47}$$

[3]Large threshold corrections at the GUT scale due to large representations of the GUT symmetry breaking sector of the theory can in principle save the GUT prediction for gauge coupling unification of non-supersymmetric theories [48].

Consider the 3×3 Yukawa matrices for down quarks and charged leptons of the form

$$Y_d = \lambda \begin{pmatrix} 0 & B & 0 \\ B & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.48}$$

$$Y_e = \lambda \begin{pmatrix} 0 & B & 0 \\ B & -3A & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.49}$$

If $B \ll A$, then the eigenvalues satisfy

$$\lambda_b = \lambda_\tau, \quad \lambda_s = \frac{1}{3}\lambda_\mu, \quad \lambda_d = 3\lambda_e. \tag{5.50}$$

Thus we now obtain the result of Eq. (5.47). Note, the relative factor of 3 in the $2-2$ term for leptons as compared to down quarks can be due to an additional Higgs in the $\overline{\mathbf{45}}$ or **75** dimensional representations of $SU(5)$ or to a Higgs in the **45** dimensional representation of $SO(10)$ (see Sect. 9).

In order to discuss neutrino masses in the context of $SU(5)$ we can easily arrange a See-Saw mechanism by adding three sterile neutrinos, $\bar{\nu}_i$, $i = 1, 2, 3$, transforming as $SU(5)$ singlets. Then we add to our Yukawa couplings the terms

$$(Y_\nu)^{ij}\ \bar{\nu}_i\ \bar{5}_{j\alpha}\ H^\alpha - \frac{1}{2}\ M^{ij}\ \bar{\nu}_i\ \bar{\nu}_j. \tag{5.51}$$

5.6 Nucleon Decay

Baryon number is necessarily violated in any GUT [56]. In $SU(5)$, nucleons decay via the exchange of gauge bosons with GUT scale masses, resulting in dimension 6 baryon number violating operators suppressed by $(1/M_G^2)$. The nucleon lifetime is calculable and given by $\tau_N \propto M_G^4/(\alpha_G^2\ m_p^5)$. The dominant decay mode of the proton (and the baryon violating decay mode of the neutron), via gauge exchange, is $p \to e^+\ \pi^0$ ($n \to e^+\ \pi^-$). In any simple gauge symmetry, with one universal GUT coupling and scale ($\alpha_G,\ M_G$), the nucleon lifetime from gauge exchange is calculable. Hence, the GUT scale may be directly observed via the extremely rare decay of the nucleon. In non-SUSY GUTs the GUT scale is of order 10^{13-15} GeV. Hence the dimension 6 baryon violating operators mediate nucleon decay with $\tau_p \ll 10^{28-32}$ years. This is ruled out by the present bounds from Super-Kamiokande, $\tau_p \geq 10^{34}$ years.

In order to get the terms in the Lagrangian which involve **X** and **Y** gauge bosons, we will use the decomposition of the relevant representations under the SM gauge group.[4]

$$\bar{\mathbf{5}} \to (\bar{\mathbf{3}}, 1)_{-2/3} + (1, \bar{\mathbf{2}})_{-1}, \tag{5.52}$$

$$\mathbf{10} \to (\mathbf{3}, \mathbf{2})_{1/3} + (\bar{\mathbf{3}}, 1)_{-4/3} + (1, 1)_{2}, \tag{5.53}$$

$$\mathbf{24} \to (\mathbf{8}, 1)_0 + (1, \mathbf{3})_0 + (1, 1)_0 + (\mathbf{3}, \bar{\mathbf{2}})_{-5/3} + (\bar{\mathbf{3}}, \mathbf{2})_{5/3}. \tag{5.54}$$

The only renormalizable gauge invariant couplings that involve the adjoint representation are

$$\mathscr{L} \supset \bar{\mathbf{5}}\ \ \mathbf{24}\ \ \mathbf{5} + \overline{\mathbf{10}}\ \ \mathbf{24}\ \ \mathbf{10}. \tag{5.55}$$

Then we can decompose these terms into their SM representations, and select the interesting terms. From

$$\begin{aligned} \bar{\mathbf{5}}\ \ \mathbf{24}\ \ \mathbf{5} \to & \left[(\bar{\mathbf{3}}, 1)_{2/3} + (1, \mathbf{2})_{-1}\right] \times \left[\cdots + (\mathbf{3}, \bar{\mathbf{2}})_{-5/3} + (\bar{\mathbf{3}}, \mathbf{2})_{5/3}\right] \\ & \times \left[(\mathbf{3}, 1)_{-2/3} + (1, \mathbf{2})_{1}\right] \end{aligned} \tag{5.56}$$

we see that the only gauge invariant combination is

$$(\bar{\mathbf{3}}, 1)_{2/3} \times (\mathbf{3}, \bar{\mathbf{2}})_{-5/3} \times (1, \mathbf{2})_1 + \text{h.c.} \tag{5.57}$$

Likewise, the $\overline{\mathbf{10}}\ \ \mathbf{24}\ \ \mathbf{10}$ term gives us

$$(\bar{\mathbf{3}}, 1)_{-4/3} \times (\bar{\mathbf{3}}, \mathbf{2})_{5/3} \times (\bar{\mathbf{3}}, \bar{\mathbf{2}})_{-1/3} + (\mathbf{3}, \mathbf{2})_{1/3} \times (\bar{\mathbf{3}}, \mathbf{2})_{5/3} \times (1, 1)_{-2} + \text{h.c.} \tag{5.58}$$

We can also see this by explicitly writing out the couplings in terms of matrices. The fermion representations are given by

$$\bar{\mathbf{5}} = \begin{pmatrix} \bar{d}_1 \\ \bar{d}_2 \\ \bar{d}_3 \\ -e \\ \nu \end{pmatrix}, \quad \mathbf{10} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \bar{u}_3 & -\bar{u}_2 & u_1 & d_1 \\ -\bar{u}_3 & 0 & \bar{u}_1 & u_2 & d_2 \\ \bar{u}_2 & -\bar{u}_1 & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & \bar{e} \\ -d_1 & -d_2 & -d_3 & -\bar{e} & 0 \end{pmatrix} \tag{5.59}$$

[4]See for example, Slansky [57], p. 96, Table 30.

The factor of $\frac{1}{\sqrt{2}}$ in front of the **10** gives us a canonically normalized kinetic term. The adjoint looks like

$$\mathbf{24} = \begin{pmatrix} & & & \mathbf{X}_1 & \mathbf{Y}_1 \\ & & & \mathbf{X}_2 & \mathbf{Y}_2 \\ & & & \mathbf{X}_3 & \mathbf{Y}_3 \\ \mathbf{X}_1^\dagger & \mathbf{X}_2^\dagger & \mathbf{X}_3^\dagger & & \\ \mathbf{Y}_1^\dagger & \mathbf{Y}_2^\dagger & \mathbf{Y}_3^\dagger & & \end{pmatrix}, \tag{5.60}$$

where we have only written down the **X** and **Y** part.

From Eq. (5.57), we have

$$(\bar{\mathbf{3}}, 1)_{2/3} \times (\mathbf{3}, \bar{\mathbf{2}})_{-5/3} \times (1, \mathbf{2})_1 \sim e^* \mathbf{X}_a^\mu \bar{d}_a - \nu^* \mathbf{Y}_a^\mu \bar{d}_a. \tag{5.61}$$

The other terms come from Eq. (5.58), but we have to be careful. We need to make sure that we antisymmetrize over all of the color indices in the $(\bar{\mathbf{3}}, 1)_{-4/3} \times (\bar{\mathbf{3}}, \mathbf{2})_{5/3} \times (\bar{\mathbf{3}}, \mathbf{2})_{-1/3}$ case:

$$(\bar{\mathbf{3}}, 1)_{-4/3} \times (\bar{\mathbf{3}}, \mathbf{2})_{5/3} \times (\bar{\mathbf{3}}, \bar{\mathbf{2}})_{-1/3} \sim \epsilon^{abc} \left[\bar{u}_a \mathbf{X}_b^{\mu\dagger} d_c^* - \bar{u}_a \mathbf{Y}_b^{\mu\dagger} u_c^* \right]. \tag{5.62}$$

Finally from $(\mathbf{3}, \mathbf{2})_{1/3} \times (\bar{\mathbf{3}}, \mathbf{2})_{5/3} \times (1, 1)_{-2}$ we have

$$(\mathbf{3}, \mathbf{2})_{1/3} \times (\bar{\mathbf{3}}, \mathbf{2})_{5/3} \times (1, 1)_{-2} \sim (\bar{e})^* \mathbf{Y}_a^{\mu\dagger} u_a - (\bar{e})^* \mathbf{X}_a^{\mu\dagger} d_a \tag{5.63}$$

Now it's easy to write down the charged currents to which the bosons couple. Note that a, b, c are SU(3) indices, and i, j are SU(2) indices:

$$\begin{aligned} J_{ai}^\mu &= (l_i)^* \bar{\sigma}^\mu \bar{d}_a + \epsilon_{abc} \epsilon_{ij} (\bar{u}_b)^* \bar{\sigma}^\mu q_j^c + \epsilon_{ij} (q_j^a)^* \bar{\sigma}^\mu \bar{e}, \\ \Rightarrow J^\mu &= (l)^* \bar{\sigma}^\mu \bar{d} + (\bar{u})^* \bar{\sigma}^\mu q + (q)^* \bar{\sigma}^\mu \bar{e}, \end{aligned} \tag{5.64}$$

where color and SU(2) indices are understood in the last line and we define the iso-doublet gauge field

$$(X^\mu)_a^i = \begin{pmatrix} X_a^\mu \\ Y_a^\mu \end{pmatrix}. \tag{5.65}$$

The effective lagrangian coupling the gauge bosons to the charged currents is then given by

$$\mathscr{L}_{cc} = \frac{g_{\text{GUT}}}{\sqrt{2}} \mathbf{X}_\mu J^\mu + h.c.. \tag{5.66}$$

Any (tree level) process that we'll be interested in will involve the exchange of an **X** or **Y** gauge boson. As in the Fermi theory, we will make the substitution for the propagator

$$\frac{-1}{p^2 - M_X^2} = \frac{1}{M_{\mathbf{X}}^2} \frac{1}{1 - \frac{p^2}{M_{\mathbf{X}}^2}} = \frac{1}{M_{\mathbf{X}}^2} \left\{ 1 + \mathscr{O}\left(\frac{p^2}{M_{\mathbf{X}}^2} \right) \right\} . \tag{5.67}$$

Integrating out the X boson leaves us with

$$\mathscr{L}_{eff} = \frac{g_{\text{GUT}}^2}{2M_{\mathbf{X}}^2} J^\mu J_\mu^* . \tag{5.68}$$

In Fig. 5.2 we show a representative Feynman diagram describing the decay of a proton to two different possible final states. Replacing the spectator up quark by a down quark, one obtains a baryon and lepton number violating neutron decay. Note, in order to calculate the nucleon decay rate one must first renormalize the dimension 6 operator down to the weak scale. Apply all weak scale threshold corrections and then further renormalize down to the nucleon mass scale. Finally, one uses chiral Lagrangian analysis to evaluate the expectation value of the three quark operator between the hadronic vacuum and the nucleon.

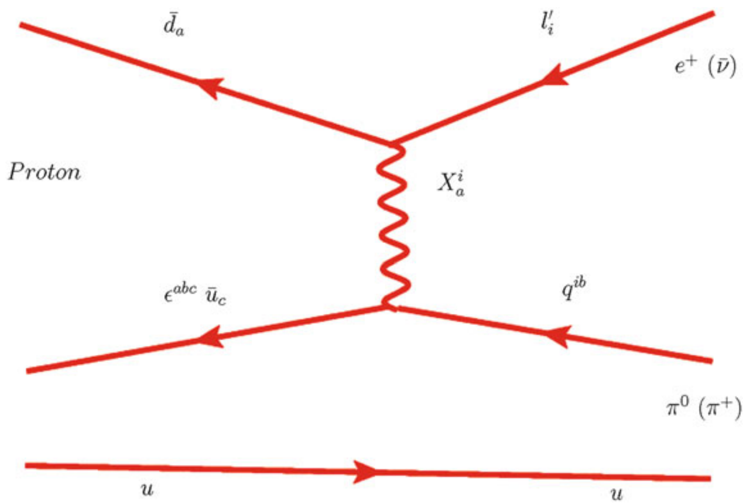

Fig. 5.2 Proton decay via dimension 6 operators

5.7 Introduction to $SO(10)$

For a nice review of $SO(10)$ group theory, see for example, the last two chapters of Georgi, "Lie Algebras in Particle Physics" [58]. The defining representation is a ten dimensional vector denoted by $\mathbf{10}_i, \quad i = 1, \cdots, 10$. $SO(10)$ is defined by the set of real orthogonal transformations $O_{ij} : O^T O = 1$ such that $\mathbf{10}'_i = O_{ij}\mathbf{10}_j$. Infinitesimal $SO(10)$ rotations are given by $O = 1 + i\tilde{\omega}$ with $\tilde{\omega}^T = -\tilde{\omega}$. We can always express the $\mathbf{10} \times \mathbf{10}$ antisymmetric matrix $\tilde{\omega}$ in the canonical form $\tilde{\omega}_{ij} \equiv \omega_{ab} \Sigma^{ab}_{ij}$. ω_{ab} are 45 real infinitesimal parameters satisfying $\omega_{ab} = -\omega_{ba}$ and $\Sigma^{ab}_{ij} = i(\delta^a_i \delta^b_j - \delta^a_j \delta^b_i)$ are the 45 generators of $SO(10)$ in the 10 dimensional representation. Note that the antisymmetric tensor product $(\mathbf{10} \times \mathbf{10})_A \equiv \mathbf{45}$ is the adjoint representation.

The $SO(10)$ generators satisfy the Lie algebra

$$[\Sigma^{ab}, \ \Sigma^{cd}]_{ik} \equiv \Sigma^{ab}_{ij} \Sigma^{cd}_{jk} - \Sigma^{cd}_{ij} \Sigma^{ab}_{jk} = [\Sigma^{ad}_{ik} \delta_{bc} - \Sigma^{ac}_{ik} \delta_{bd} + \Sigma^{bc}_{ik} \delta_{ad} - \Sigma^{bd}_{ik} \delta_{ac}]. \tag{5.69}$$

The adjoint representation transforms as follows : $\mathbf{45}'_{ij} = O_{ik} O_{jl} \mathbf{45}_{kl}$ or $\mathbf{45}'_{ij} = (O\, \mathbf{45}\, O^T)_{ij}$.

In general the tensor product $(\mathbf{10} \times \mathbf{10}) = (\mathbf{10} \times \mathbf{10})_A + (\mathbf{10} \times \mathbf{10})_S = \mathbf{45} + \mathbf{54} + \mathbf{1}$. The 54 dimensional representation is denoted by the symmetric tensor $\mathbf{54}_{ij} = \mathbf{54}_{ji}, \ Tr(\mathbf{54}) = \mathbf{0}$ with transformations $\mathbf{54}' = O\, \mathbf{54}\, O^T$.

The spinor representation of $SO(10)$ can be defined in terms of $2^5 \times 2^5$ dimensional representations of a Clifford algebra $\Gamma^a, \ a = 1, \cdots, 10$, just as for example the spinor representation of SO(4) is represented in terms of 4×4 Dirac gamma matrices. The Γs can be given by a tensor product of five Pauli matrices—

$$\begin{aligned}
\Gamma_1 &= \sigma^1_2 \sigma^2_3 \sigma^3_3 \sigma^4_3 \sigma^5_3 & \Gamma_2 &= -\sigma^1_1 \sigma^2_3 \sigma^3_3 \sigma^4_3 \sigma^5_3 \\
\Gamma_3 &= \mathbb{1}^1 \sigma^2_2 \sigma^3_3 \sigma^4_3 \sigma^5_3 & \Gamma_4 &= -\mathbb{1}^1 \sigma^1_1 \sigma^3_3 \sigma^4_3 \sigma^5_3 \\
\Gamma_5 &= \mathbb{1}^1 \mathbb{1}^2 \sigma^3_2 \sigma^4_3 \sigma^5_3 & \Gamma_6 &= -\mathbb{1}^1 \mathbb{1}^2 \sigma^3_1 \sigma^4_3 \sigma^5_3 \\
\Gamma_7 &= \mathbb{1}^1 \mathbb{1}^2 \mathbb{1}^3 \sigma^4_2 \sigma^5_3 & \Gamma_8 &= -\mathbb{1}^1 \mathbb{1}^2 \mathbb{1}^3 \sigma^4_1 \sigma^5_3 \\
\Gamma_9 &= \mathbb{1}^1 \mathbb{1}^2 \mathbb{1}^3 \mathbb{1}^4 \sigma^5_2 & \Gamma_{10} &= -\mathbb{1}^1 \mathbb{1}^2 \mathbb{1}^3 \mathbb{1}^4 \sigma^5_1
\end{aligned} \tag{5.70}$$

satisfy $\Gamma^{a\dagger} = \Gamma^a, \quad \{\Gamma^a, \ \Gamma^b\} = 2\delta^{ab}$. We can also define $\Gamma^{11} \equiv (-i)^5 \prod_{a=1}^{10} \Gamma^a = \prod_{j=1}^{5} \sigma^j_3$ satisfying $\{\Gamma^{11}, \ \Gamma^a\} = 0$ for all a. The generators of $SO(10)$ in the spinor representation are now given by

$$\Sigma^{ab} = \frac{i}{4}[\Gamma^a, \ \Gamma^b]. \tag{5.71}$$

Note $[\Gamma^{11}, \ \Sigma^{ab}] = 0$ and $(\Gamma^{11})^2 = 1$. Hence Γ^{11} has eigenvalues ± 1 which divides the 32 dimensional spinor into two irreducible representations of $SO(10)$ which are the $\mathbf{16}$ and $\overline{\mathbf{16}}$ spinor representations.

In order to generate some intuition on how $SO(10)$ acts on the spinor representations, we use the gamma matrices to define operators satisfying a Heisenberg algebra of creation and annihilation operators. Let

$$A_\alpha = \frac{\Gamma^{2\alpha-1} + i\Gamma^{2\alpha}}{2}, \quad \alpha = 1, \cdots, 5 \tag{5.72}$$

and

$$A_\alpha^\dagger = \frac{\Gamma^{2\alpha-1} - i\Gamma^{2\alpha}}{2}. \tag{5.73}$$

The As satisfy $\{A_\alpha,\ A_\beta\} = 0, \quad \{A_\alpha,\ A_\beta^\dagger\} = \delta_{\alpha\beta}$. We could now rewrite the generators of $SO(10)$ explicitly in terms of products of As and $A^\dagger$s. Instead of doing this let me directly identify an $SU(5)$ subgroup of $SO(10)$. In fact the set of generators $\{\Sigma^{ab}\}$ are equivalent to the set of generators $\{Q_A,\ \Delta_{\alpha\beta},\ \Delta_{\alpha\beta}^\dagger,\ X\}$ defined by

$$Q_A = A_\alpha^\dagger \frac{\lambda_{\alpha\beta}^A}{2} A_\beta, \quad A = 1, \cdots, 2 \tag{5.74}$$

where $\lambda_{\alpha\beta}^A$ are the 5×5 traceless hermitian generators of $SU(5)$ in the 5 dimensional representation. It is then easy to see that the Qs satisfy the Lie algebra of $SU(5)$, $[Q_A,\ Q_B] = if_{ABC}\, Q_C$. Define

$$\Delta_{\alpha\beta} = A_\alpha A_\beta = -\Delta_{\beta\alpha}, \quad \Delta_{\alpha\beta}^\dagger = A_\beta^\dagger A_\alpha^\dagger = -\Delta_{\beta\alpha}^\dagger. \tag{5.75}$$

Finally, we define

$$X = -2 \sum_{\alpha=1}^{5} (A_\alpha^\dagger A_\alpha - \frac{1}{2}), \tag{5.76}$$

the $U(1)$ generator which commutes with the generators of $SU(5)$.

Let us now define the **16**, $\overline{\mathbf{16}}$ representations explicitly. Consider first the **16** which contains a $\mathbf{10} + \bar{\mathbf{5}} + \mathbf{1}$ under $SU(5)$. Let $|0\rangle \equiv |\mathbf{1}\rangle \equiv [0]$ be the $SU(5)$ invariant state contained in the **16**, such that $Q_A|0\rangle \equiv 0$. It is thus the vacuum state for the annihilation operators A_α (i.e. $A_\alpha|0\rangle \equiv 0$). It is thus an $SU(5)$ singlet and also a zero index tensor under $SU(5)$ transformations. We now have $\Delta_{\alpha\beta}^\dagger|0\rangle = |\mathbf{10}\rangle^{\alpha\beta} = [2]$, i.e. a 2 index antisymmetric tensor or a **10** under $SU(5)$. Finally, $\epsilon_{\alpha\beta\gamma\delta\lambda}\Delta_{\alpha\beta}^\dagger\Delta_{\gamma\delta}^\dagger|0\rangle = |\bar{\mathbf{5}}\rangle_\lambda = [4]$. Thus, in summary, we have defined the $\mathbf{16} = \mathbf{10} + \bar{\mathbf{5}} + \mathbf{1}$ by

$$|\mathbf{1}\rangle = |0\rangle, \quad |\mathbf{10}\rangle^{\alpha\beta} = \Delta_{\alpha\beta}^\dagger|0\rangle, \quad |\bar{\mathbf{5}}\rangle_\lambda = \epsilon_{\alpha\beta\gamma\delta\lambda}\Delta^{\dagger^{\alpha\beta}}\Delta^{\dagger^{\gamma\delta}}|0\rangle. \tag{5.77}$$

Similarly the $\overline{\mathbf{16}} = \overline{\mathbf{10}} + \mathbf{5} + \mathbf{1}$ is defined by

$$|\mathbf{5}\rangle^{\alpha} = A^{\dagger}_{\alpha}|0\rangle, \;\; |\overline{\mathbf{10}}\rangle_{\delta\rho} = \epsilon_{\alpha\beta\gamma\delta\rho}\Delta^{\dagger}_{\alpha\beta}A^{\dagger}_{\gamma}|0\rangle, \;\; |\mathbf{1}\rangle = \epsilon_{\alpha\beta\gamma\delta\rho}\Delta^{\dagger}_{\alpha\beta}\Delta^{\dagger}_{\gamma\delta}A^{\dagger}_{\rho}|0\rangle. \tag{5.78}$$

$SO(10)$ is a rank 5 group, meaning there are 5 $U(1)$ generators in the Cartan subalgebra. The five generators can be defined as:

$$\begin{aligned}
\Sigma^{12} &= \frac{i}{4}[\Gamma^1,\ \Gamma^2] \equiv (A^{\dagger}_1 A_1 - 1/2), \qquad (5.79)\\
\Sigma^{34} &= \frac{i}{4}[\Gamma^3,\ \Gamma^4] \equiv (A^{\dagger}_2 A_2 - 1/2),\\
\Sigma^{56} &= \frac{i}{4}[\Gamma^5,\ \Gamma^6] \equiv (A^{\dagger}_3 A_3 - 1/2),\\
\Sigma^{78} &= \frac{i}{4}[\Gamma^7,\ \Gamma^8] \equiv (A^{\dagger}_4 A_4 - 1/2),\\
\Sigma^{9\,10} &= \frac{i}{4}[\Gamma^9,\ \Gamma^{10}] \equiv (A^{\dagger}_5 A_5 - 1/2).
\end{aligned}$$

The first 3 act on color indices and the last two act on weak indices. Thus the $SU(5)$ invariant $U(1)$ generator in the 16 dimensional representation is given by

$$X = -2\sum_{\alpha=1}^{5}(A^{\dagger}_{\alpha}A_{\alpha} - 1/2) = -2(\Sigma^{12} + \Sigma^{34} + \Sigma^{56} + \Sigma^{78} + \Sigma^{910}). \tag{5.80}$$

The 10 dimensional representation can be expressed in terms of a $(5\times5)\otimes(2\times2)$ tensor product notation. We can use the above formula to write an expression for X in this basis. We find

$$X = 2x \otimes \eta$$

where

$$x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\eta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2. \tag{5.81}$$

Similarly we can identify the other $U(1)$s which commute with $SU(3) \times SU(2) \times U(1)_Y$:

$$Y = -\frac{2}{3}\sum_{\alpha=1}^{3}(A_\alpha^\dagger A_\alpha - 1/2) + \sum_{\alpha=4}^{5}(A_\alpha^\dagger A_\alpha - 1/2)|_{on16} = y \otimes \eta|_{on10} \tag{5.82}$$

where $y = \begin{pmatrix} 2/3 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$;

$$B - L = -\frac{2}{3}\sum_{\alpha=1}^{3}(A_\alpha^\dagger A_\alpha - 1/2)|_{on16} = \frac{2}{3}(b - l) \otimes \eta \tag{5.83}$$

where $(b - l) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$; and

$$T_{3R} = -\frac{1}{2}\sum_{\alpha=4}^{5}(A_\alpha^\dagger A_\alpha - 1/2)|_{on16} = \frac{1}{2}t_{3R} \otimes \eta \tag{5.84}$$

where $t_{3R} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

It is a useful exercise to use the definition of the **16** defined above and the definition of Y in terms of number operators to identify the hypercharge assignments of the states in the **16**.

Note that we will use fields in the adjoint (**45**) representation to break $SO(10)$ to the SM. A **45** vev in the X direction will break $SO(10)$ to $SU(5) \times U(1)_X$. The vev of a $\mathbf{16} + \overline{\mathbf{16}}$ in the $\overline{\nu}$ directions can then break X leaving $SU(5)$ invariant. We could then use a 45 with vev in either the $Y, B - L$ or T_{3R} directions to break $SU(5)$ to $SU(3) \times SU(2) \times U(1)_Y$. Note also that either (X, Y) or $(B - L, T_{3R})$ span the 2 dimensional space of $U(1)$s which commute with $SU(3) \times SU(2) \times U(1)_Y$.

Finally, in an $SO(10)$ SUSY GUT the **16** of $SO(10)$ contains one family of fermions and their supersymmetric partners. The **10** of $SO(10)$ contains a pair of Higgs doublets necessary to do the electroweak symmetry breaking. Under $SU(5)$

we have $\mathbf{10} = \mathbf{5} + \bar{\mathbf{5}}$. The simplest dimension 4 Yukawa coupling of the electroweak Higgs to a single family (consider the third generation) is given by

$$\lambda\ \mathbf{16}_3\ \mathbf{10}\ \mathbf{16}_3. \tag{5.85}$$

The $SO(10)$ symmetry relation which follows is

$$\lambda_t = \lambda_b = \lambda_\tau = \lambda_{\nu_\tau} = \lambda. \tag{5.86}$$

For example, see [59–61] (for SUSY $SO(10)$ see [62–71]).

Chapter 6
SUSY GUTs

In order to construct a supersymmetric grand unified theory, we need to convert all the fields of the previous lecture into superfields. For $SU(5)$ we have the chiral superfields containing the SM fermions and Higgses,

$$\bar{\mathbf{5}}_i(y,\theta), \quad \mathbf{10}_i(y,\theta), \quad \bar{N}_i(y,\theta), \quad H(y,\theta), \quad \bar{H}(y,\theta) \tag{6.1}$$

with the family index, $i = 1,2,3$, and we have suppressed the $SU(5)$ indices. In addition, there are the gauge boson superfields, $V = V_A\ T_A$ and gauge field strength superfield $W_\alpha(y,\theta)$. The GUT symmetry is assumed to be broken at the unification scale. Let's first consider GUT symmetry breaking and Higgs doublet-triplet splitting, followed by a discussion of gauge coupling unification and nucleon decay. We shall see that the latter issues are coupled.

The $SU(5)$ SUSY GUT Lagrangian contains three terms.

$$\mathscr{L}_{gauge-kinetic} = [\frac{1}{8\ g_G^2} \int d^2\theta Tr(\mathscr{W}^{G\alpha}\ \mathscr{W}^G{}_\alpha) + h.c.], \tag{6.2}$$

$\mathscr{L}_{gauge-matter} = [\int d^4\theta \mathscr{K}]$ where the renormalizable Kähler potential is given by

$$\mathscr{K} = \sum_i \left[\mathbf{10}_i^\dagger\ exp[-2\ g_G\ V^G]\ \mathbf{10}_i + \bar{\mathbf{5}}_i^\dagger\ exp[-2\ g_G\ V^G]\ \bar{\mathbf{5}}_i + \bar{N}_i^\dagger\ \bar{N}_i\right] \tag{6.3}$$
$$+(H^\dagger\ exp[-2\ g_G\ V^G]\ H + \bar{H}^\dagger\ exp[-2\ g_G\ V^G]\ \bar{H}) + \cdots,$$

and the superpotential

$$\mathscr{W} \supset \frac{1}{4}\ (Y_u)^{ij}\ \mathbf{10}_i^{\alpha\beta}\ \mathbf{10}_j^{\gamma\delta}\ H^\eta\ \epsilon_{\alpha\beta\gamma\delta\eta} + (Y_d)^{ij}\ \bar{H}_\alpha\ \mathbf{10}_i^{\alpha\beta}\ \bar{\mathbf{5}}_{j\beta} + (Y_\nu)^{ij}\ \bar{N}_i\ H^\alpha\ \bar{\mathbf{5}}_{j\alpha} - \frac{1}{2} M_{ij}\ \bar{N}_i\ \bar{N}_j. \tag{6.4}$$

Note, the superpotential is of the same form as in Eqs. (5.45) and (5.51).

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6.1 GUT Symmetry Breaking

Consider first $SU(5)$ where the simplest GUT breaking sector was proposed [29, 72]. A chiral superfield, Σ, in the adjoint representation of $SU(5)$ is introduced with superpotential

$$\mathscr{W} = \frac{\lambda}{3}\, Tr(\Sigma)^3 - \frac{M}{2}\, Tr(\Sigma)^2 + \bar{H}\,(\lambda'\,\Sigma\, - M')\, H. \tag{6.5}$$

The solution to the F-term equations for Σ are given by

$$-F^*_\Sigma = \lambda\,(\Sigma^2 - \frac{1}{5} Tr(\Sigma)^2\, \mathbb{1}) - M\,\Sigma = 0, \tag{6.6}$$

is

$$\Sigma = \frac{\sqrt{60}}{\lambda}\, M\, T_{24}. \tag{6.7}$$

This solution provides a supersymmetric solution with $D_A = 0$ for all A which breaks $SU(5) \to SU(3) \times SU(2) \times U(1)_Y$.[1] Then if we choose

$$M' = 3\frac{\lambda'}{\lambda}\, M, \tag{6.8}$$

the Higgs doublets are massless while the color triplet Higgses have a supersymmetric mass term given by the term in the superpotential, $-5\,\frac{\lambda'}{\lambda}\, M\, \bar{T}\, T$. It is easy to see that only the states in the MSSM are present in the low energy theory below the GUT scale of order M/λ. The breaking of $SU(5)$ to the SM gauge group is quite natural. However, splitting the Higgs doublets and triplets requires fine-tuning parameters in the Lagrangian, in particular, see the relation in Eq. (6.8). Unfortunately this fine-tuning spoils the solution to the gauge hierarchy problem or why $M_Z << M_{GUT}$. This is another manifestation of the μ problem discussed earlier.

There are "natural" solutions to the problem of doublet-triplet splitting [73, 74]; the so-called missing doublet mechanism in $SU(5)$ and the missing vev mechanism in $SO(10)$. We will save the discussion of these mechanisms and also the breaking of $SO(10)$ for later. For now let us discuss gauge coupling unification.

[1] Note, there are other solutions which break $SU(5)$ to $SU(4) \times U(1)$ or keep $SU(5)$ unbroken.

6.2 Gauge Coupling Unification

Now let us consider gauge coupling unification in the context of a supersymmetric theory. The only difference is that now we need to include all the new superpartners in the RG equations below the GUT scale and above the supersymmetry breaking scale. The general one loop RG equations are found in Eq. (5.36). For N = 1 supersymmetric theories, Eq. (5.36) can be made more compact. We have

$$b_i = 3C_2(G_i) - T_R\, N_\chi \tag{6.9}$$

where the first term takes into account the vector multiplets and N_χ is the number of left-handed chiral multiplets in the representation R [75]. The solution to the one loop RG equations is given by

$$\alpha_i(M_Z)^{-1} = \alpha_G^{-1} - \frac{b_i}{2\pi} \ln(\frac{M_G}{M_Z}). \tag{6.10}$$

For SUSY we have

$$\mathbf{b}_{SUSY} = (-2N_{fam} - \frac{3}{5}N_{(H_u+H_d)}, 6 - 2N_{fam} - N_{(H_u+H_d)}, 9 - 2N_{fam}) \tag{6.11}$$

where $N_{(H_u+H_d)}$ is the number of pairs of Higgs doublets. Thus for the MSSM we have [75]

$$\mathbf{b}^{MSSM} = (-33/5, -1, 3). \tag{6.12}$$

The one loop RG equations can be solved analytically for α_G, M_G, and a prediction for $\alpha_s(M_Z)$ [see Eq. (5.4)]. Given the experimental values $\sin^2\theta_W(M_Z) \approx 0.23$ and $\alpha_{EM}(M_Z)^{-1} \approx 128$ we find $M_G \approx 2.7 \times 10^{16}$ GeV, $\alpha_G^{-1} \approx 24$ and the predicted strong coupling $\alpha_3(M_Z) \approx 0.12$. How well does this agree with the data? According to the PDG the average value of $\alpha_s(M_Z) = 0.1185 \pm 0.0006$ [47].[2] So at one loop the MSSM is quite good, while non-SUSY GUTs were clearly excluded (see Fig. 5.1) [29, 51, 72, 75–77].[3]

Presently, the MSSM is compared to data using 2 loop RG running from the weak to the GUT scale with one loop threshold corrections included at the weak scale. These latter corrections have small contributions from integrating out the W, Z, and top quark. But the major contribution comes from integrating out the

[2] See problem 6.

[3] The experimental evidence for gauge coupling unification in SUSY GUTs was first recognized in 1991 [78–81].

presumed SUSY spectrum. With a "typical" SUSY spectrum[4] and assuming no threshold corrections at the GUT scale, one finds a value for $\alpha_s(M_Z) \geq 0.127$ which is too large [82]. It is easy to see where this comes from using the approximate analytic formula

$$\alpha_i^{-1}(M_Z) = \alpha_G^{-1} - \frac{b_i^{MSSM}}{2\pi} \ln(\frac{M_G}{M_Z}) + \delta_i \tag{6.13}$$

where

$$\delta_i = \delta_i^h + \delta_i^2 + \delta_i^l. \tag{6.14}$$

The constants δ_i^2, δ_i^l, δ_i^h represent the 2 loop running effects [51, 77], the weak scale threshold corrections and the GUT scale threshold corrections, respectively. We have

$$\delta_i^2 \approx -\frac{1}{\pi} \sum_{j=1}^{3} \frac{b_{ij}^{MSSM}}{b_j^{MSSM}} \log \left[1 - b_j^{MSSM} \left(\frac{3 - 8\sin^2\theta_W}{36\sin^2\theta_W - 3} \right) \right] \tag{6.15}$$

where the matrix b_{ij}^{MSSM} is given by Einhorn and Jones [51], Marciano and Senjanovic [77]

$$b_{ij}^{MSSM} = \begin{pmatrix} \frac{199}{100} & \frac{27}{20} & \frac{22}{5} \\ \frac{9}{20} & \frac{25}{4} & 6 \\ \frac{11}{20} & \frac{9}{4} & \frac{7}{2} \end{pmatrix}. \tag{6.16}$$

The light thresholds are given by

$$\delta_i^l = \frac{1}{\pi} \sum_j b_i^l(j) \log(\frac{m_j}{M_Z}) \tag{6.17}$$

where the sum runs over all states at the weak scale including the top, W, Higgs and the supersymmetric spectrum. Finally the GUT scale threshold correction is given by

$$\delta_i^h = -\frac{1}{2\pi} \sum_\zeta b_i^\zeta \log(\frac{M_\zeta}{M_G}). \tag{6.18}$$

[4]By a typical SUSY spectrum we mean one determined by CMSSM type GUT scale boundary conditions with universal scalar masses, universal gaugino masses and a universal A parameter.

In general the prediction for $\alpha_3(M_Z)$ is given by

$$\begin{aligned}
\alpha_3^{-1}(M_Z) &= (\frac{b_3 - b_1}{b_2 - b_1})\alpha_2^{-1}(M_Z) - (\frac{b_3 - b_2}{b_2 - b_1})\alpha_1^{-1}(M_Z) \\
&\quad + (\frac{b_3 - b_2}{b_2 - b_1})\delta_1 - (\frac{b_3 - b_1}{b_2 - b_1})\delta_2 + \delta_3 \\
&= \frac{12}{7}\alpha_2^{-1}(M_Z) - \frac{5}{7}\alpha_1^{-1}(M_Z) + \frac{1}{7}(5\delta_1 - 12\delta_2 + 7\delta_3) \\
&\equiv (\alpha_3^{LO})^{-1} + \delta_s
\end{aligned} \tag{6.19}$$

where $b_i \equiv b_i^{MSSM}$, $\alpha_3^{LO}(M_Z)$ is the leading order one-loop result and $\delta_s \equiv \frac{1}{7}(5\delta_1 - 12\delta_2 + 7\delta_3)$. We find $\delta_s^2 \approx -0.82$ [83] and $\delta_s^l = -0.04 + \frac{19}{28\pi}\ln(\frac{T_{SUSY}}{M_Z})$ where the first term takes into account the contribution of the W, top and the correction from switching from the $\overline{MS}$ to $\overline{DR}$ RG schemes and (following [84])

$$T_{SUSY} = m_{\tilde{H}}(\frac{m_{\tilde{W}}}{m_{\tilde{g}}})^{28/19}\left[(\frac{m_{\tilde{l}}}{m_{\tilde{q}}})^{3/19}(\frac{m_H}{m_{\tilde{H}}})^{3/19}(\frac{m_{\tilde{W}}}{m_{\tilde{H}}})^{4/19}\right]. \tag{6.20}$$

For a Higgsino mass $m_{\tilde{H}} = 400\,\text{GeV}$, a Wino mass $m_{\tilde{W}} = 300\,\text{GeV}$, a gluino mass $m_{\tilde{g}} = 900\,\text{GeV}$ and all other mass ratios of order one, we find $\delta_s^l \approx -0.12$. If we assume $\delta_s^h = 0$, we find the predicted value of $\alpha_3(M_Z) = 0.135$. In order to obtain a reasonable value of $\alpha_3(M_Z)$ with only weak scale threshold corrections, we need $\delta_s^2 + \delta_s^l \approx 0$ corresponding to a value of $T_{SUSY} \sim 5\,\text{TeV}$. But this is very difficult considering the weak dependence T_{SUSY} [Eq. (6.20)] has on squark and slepton masses. Thus in order to have $\delta_s \approx 0$ we need a GUT scale threshold correction

$$\delta_s^h \approx +0.94. \tag{6.21}$$

At the GUT scale we have

$$\alpha_i^{-1}(M_G) = \alpha_G^{-1} + \delta_i^h. \tag{6.22}$$

Define

$$\tilde{\alpha}_G^{-1} = \frac{1}{7}[12\alpha_2^{-1}(M_G) - 5\alpha_1^{-1}(M_G)] \tag{6.23}$$

(or if the GUT scale is defined at the point where α_1 and α_2 intersect, then $\tilde{\alpha}_G \equiv \alpha_1(M_G) = \alpha_2(M_G)$. Hence, in order to fit the data, we need a GUT threshold correction

$$\epsilon_3 \equiv \frac{\alpha_3(M_G) - \tilde{\alpha}_G}{\tilde{\alpha}_G} = -\tilde{\alpha}_G\,\delta_s^h \approx -4\%. \tag{6.24}$$

Note, when performing an exact two loop RG analysis of gauge coupling unification starting with CMSSM boundary conditions at the GUT scale, one obtains the same result. Once again, in order to fit the low energy values of $\alpha_i(M_Z), \quad i = 1, 2, 3$ one needs a value of $\epsilon_3 \approx -4\%$.[5]

6.3 Nucleon Decay

In SUSY $SU(5)$, gauge bosons contribute to nucleon decay just as in non-SUSY $SU(5)$. The main difference is in the value of α_G and M_G. In SUSY GUTs, the GUT scale is of order 3×10^{16} GeV, as compared to the GUT scale in non-SUSY GUTs which is of order 10^{15} GeV. Hence the dimension 6 baryon violating operators are significantly suppressed in SUSY GUTs [29, 72, 75, 76] with $\tau_p \sim 10^{34-38}$ years.

However, in SUSY GUTs there are additional sources for baryon number violation—dimension 4 and 5 operators [85, 86]. Although our notation does not change, when discussing SUSY GUTs all fields are implicitly bosonic superfields and the operators considered are the so-called F terms which contain two fermionic components and the rest scalars or products of scalars. Within the context of $SU(5)$ the dimension 4 and 5 operators have the form $(\mathbf{10}\ \bar{\mathbf{5}}\ \bar{\mathbf{5}}) \supset (\bar{U}\ \bar{D}\ \bar{D}) + (Q\ L\ \bar{D}) + (\bar{E}\ L\ L)$ and $(\mathbf{10}\ \mathbf{10}\ \mathbf{10}\ \bar{\mathbf{5}}) \supset (Q\ Q\ Q\ L) + (\bar{U}\ \bar{U}\ \bar{D}\ \bar{E}) + B$ and L conserving terms, respectively. The dimension 4 operators are renormalizable with dimensionless couplings; similar to Yukawa couplings. On the other hand, the dimension 5 operators have a dimensionful coupling of order $(1/M_G)$.

The dimension 4 operators violate baryon number or lepton number, respectively, but not both. The nucleon lifetime is extremely short if both types of dimension 4 operators are present in the low energy theory (see Fig. 6.1). In this case if the coupling for the baryon number violating operator is λ and the lepton number violating operator is λ', then the product is constrained to satisfy $\lambda\ \lambda' < 10^{-27}$ to be consistent with nucleon decay bounds. However both types can be eliminated by requiring R parity.

In $SU(5)$ the Higgs doublets reside in a $\mathbf{5_H}$, $\bar{\mathbf{5}}_\mathbf{H}$ and R parity distinguishes the $\bar{\mathbf{5}}$ (quarks and leptons) from $\bar{\mathbf{5}}_\mathbf{H}$ (Higgs). R parity [6] (or its cousin, family reflection symmetry (see [29] and [87, 88]) takes $F \to -F, \quad H \to H$ with $F = \{\mathbf{10}, \bar{\mathbf{5}}\}, \quad H = \{\bar{\mathbf{5}}_\mathbf{H}, \mathbf{5_H}\}$. This forbids the dimension 4 operator $(\mathbf{10}\ \bar{\mathbf{5}}\ \bar{\mathbf{5}})$, but allows the Yukawa couplings of the form $(\mathbf{10}\ \bar{\mathbf{5}}\ \bar{\mathbf{5}}_\mathbf{H})$ and $(\mathbf{10}\ \mathbf{10}\ \mathbf{5_H})$. It also forbids the dimension 3, lepton number violating, operator $(\bar{\mathbf{5}}\ \mathbf{5_H}) \supset (L\ \ H_u)$ with a coefficient with dimensions of mass which, like the μ parameter, could be of order the weak scale and the dimension 5, baryon number violating, operator $(\mathbf{10}\ \mathbf{10}\ \mathbf{10}\ \bar{\mathbf{5}}_\mathbf{H}) \supset (Q\ Q\ Q\ H_d) + \cdots$.

[5]The code SOFTSUSY defines the GUT scale where $\alpha_1(M_G) = \alpha_2(M_G)$. A value of ϵ_3 can be read out and again one finds for the CMSSM, $\epsilon_3 \approx -4\%$.

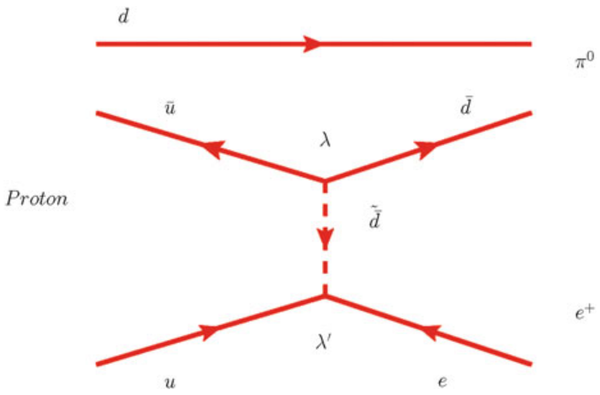

Fig. 6.1 The effective four fermi operator for proton decay obtained by having both baryon (coupling λ) and lepton (coupling λ') number violating dimension 4 operators

Note, in the MSSM it is possible to retain R parity violating operators at low energy as long as they violate either baryon number or lepton number only but not both. Such schemes are natural if one assumes a low energy symmetry, such as lepton number, baryon number or a baryon parity [89, 90]. However these symmetries cannot be embedded in a GUT. Thus, in a SUSY GUT, only R parity can prevent unwanted dimension four operators. Hence, by naturalness arguments, R parity must be a symmetry in the effective low energy theory of any SUSY GUT.[6]

Note also, R parity distinguishes Higgs multiplets from ordinary families. In $SU(5)$, Higgs and quark/lepton multiplets have identical quantum numbers; while in $E(6)$, Higgs and families are unified within the fundamental **27** representation. Only in $SO(10)$ are Higgs and ordinary families distinguished by their gauge quantum numbers. Moreover the $\mathbb{Z}_4$ center of $SO(10)$ distinguishes **10**s from **16**s and can be associated with R parity [94].

Dimension 5 baryon number violating operators may be forbidden at tree level by symmetries in $SU(5)$, etc. These symmetries are typically broken however by the VEVs responsible for the color triplet Higgs masses. Consequently these dimension 5 operators are generically generated via color triplet Higgsino exchange. Hence, the color triplet partners of Higgs doublets must necessarily obtain mass of order the GUT scale. The dominant decay modes from dimension 5 operators are $p \to K^+ \ \bar{\nu} \quad (n \to K^0 \ \bar{\nu})$. This is due to a simple symmetry argument;

[6]This does not mean to say that R parity is guaranteed to be satisfied in any GUT. For example the authors of [91, 92] use constrained matter content to selectively generate safe effective R parity violating operators in a GUT. For a review on R parity violating interactions, see [93]. In [92], the authors show how to obtain the effective R parity violating operator $O^{ijk} = (\bar{5}^j \cdot \bar{5}^k)_{\overline{15}} \cdot (10^i \cdot \Sigma)_{15}$ where Σ is an $SU(5)$ adjoint field and the subscripts $\overline{15}, 15$ indicate that the product of fields in parentheses have been projected into these $SU(5)$ directions. As a consequence the operator O^{ijk} is symmetric under interchange of the two $\bar{5}$ states, $O^{ijk} = O^{ikj}$, and out of $\mathbf{10}\ \bar{\mathbf{5}}\ \bar{\mathbf{5}}$ only the lepton number/R parity violating operator $QL\bar{D}$ survives.

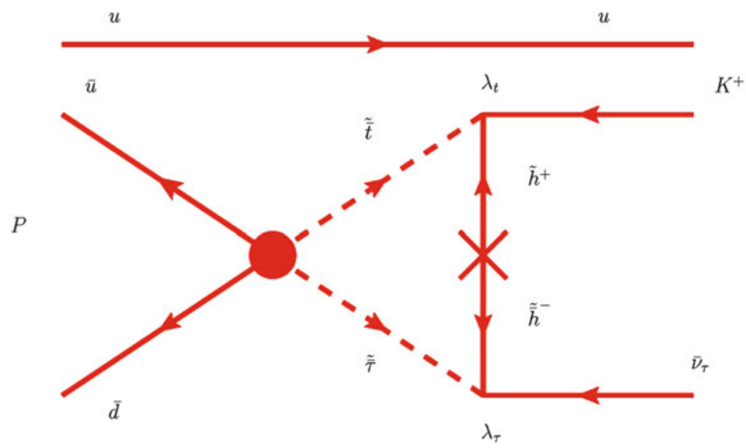

Fig. 6.2 The effective four fermi operator for proton decay obtained by integrating out sparticles at the weak scale

the operators $(Q_i\ Q_j\ Q_k\ L_l)$, $(\bar{U}_i\ \bar{U}_j\ \bar{D}_k\ \bar{E}_l)$ (where $i,\ j,\ k,\ l = 1, 2, 3$ are family indices and color and weak indices are implicit) must be invariant under $SU(3)_C$ and $SU(2)_L$. As a result their color and weak doublet indices must be antisymmetrized. However since these operators are given by bosonic superfields, they must be totally symmetric under interchange of all indices. Thus the first operator vanishes for $i = j = k$ and the second vanishes for $i = j$. Hence a second or third generation particle must appear in the final state [87, 88].

The dimension 5 operator contribution to proton decay requires a sparticle loop at the SUSY scale to reproduce an effective dimension 6 four fermi operator for proton decay (see Fig. 6.2). The loop factor is of the form

$$(LF) \propto \frac{\lambda_t\,\lambda_\tau}{16\pi^2}\frac{\sqrt{\mu^2 + M_{1/2}^2}}{m_{16}^2} \tag{6.25}$$

leading to a decay amplitude

$$A(p \to K^+\bar{\nu}) \propto \frac{c\ c}{M_T^{eff}}\ (\mathrm{LF}). \tag{6.26}$$

In any predictive SUSY GUT, the coefficients c are 3×3 matrices related to (but not identical to) Yukawa matrices. Thus these tend to suppress the proton decay amplitude. However this is typically not sufficient to be consistent with the experimental bounds on the proton lifetime. Thus it is also necessary to minimize the loop factor, (LF). This can be accomplished by taking $\mu, M_{1/2}$ *small* and m_{16} *large*. Finally the effective Higgs color triplet mass M_T^{eff} must be MAXIMIZED. With these caveats, it is possible to obtain rough theoretical bounds on the proton lifetime given by [95–99]

$$\tau_{p\to K^+\bar{\nu}} \leq (\frac{1}{3} - 3) \times 10^{34} \text{ years.} \tag{6.27}$$

6.4 Gauge Coupling Unification and Proton Decay

The dimension 5 operator (see Fig. 6.3) is given in terms of the matrices c and an effective Higgs triplet mass by

$$\frac{1}{M_T^{eff}} \left[Q\,\frac{1}{2}\,c_{qq}Q\;Q\;c_{ql}L + \overline{U}\;c_{ud}\overline{D}\;\overline{U}\;c_{ue}\overline{E} \right]. \tag{6.28}$$

Note, M_T^{eff} can be much greater than M_G without fine-tuning and without having any particle with mass greater than the GUT scale. Consider a theory with two pairs of Higgs 5_i and $\bar{5}_i$ with $i = 1, 2$ at the GUT scale with only $5_1,\ \bar{5}_1$ coupling to quarks and leptons. Then we have

$$\frac{1}{M_T^{eff}} = (M_T^{-1})_{11}. \tag{6.29}$$

If the Higgs color triplet mass matrix is given by

$$M_T = \begin{pmatrix} 0 & M_G \\ M_G & X \end{pmatrix} \tag{6.30}$$

then we have

$$\frac{1}{M_T^{eff}} \equiv \frac{X}{M_G^2}. \tag{6.31}$$

Thus for $X << M_G$ we obtain $M_T^{eff} >> M_G$. Note, however, the color triplets have mass of order M_G, yet proton decay is still suppressed.

We assume that the Higgs doublet mass matrix, on the other hand, is of the form

$$M_D = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} \tag{6.32}$$

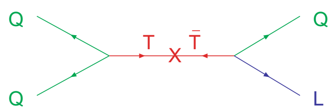

Fig. 6.3 The effective dimension 5 operator for proton decay

with two light Higgs doublets. Note this mechanism is natural in $S0(10)$ [73, 74, 100] with a superpotential of the form

$$W \supset 10\;45\;10' + X\;(10')^2 \tag{6.33}$$

with only ten coupling to quarks and leptons, X is a gauge singlet and $\langle 45 \rangle = (B-L)\,M_G$.

So we can suppress nucleon decay rates, however, we shall now show that this is at the expense of adversely affecting gauge coupling unification. Recall $\epsilon_3 \equiv \frac{(\alpha_3(M_G)-\tilde{\alpha}_G)}{\tilde{\alpha}_G} \sim -4\%$. At one loop we find

$$\epsilon_3 = \epsilon_3^{\text{Higgs}} + \epsilon_3^{\text{GUT breaking}} + \cdots . \tag{6.34}$$

Moreover [95]

$$\epsilon_3^{\text{Higgs}} = \frac{3\alpha_G}{5\pi} \ln(\frac{M_T^{eff}}{M_G}). \tag{6.35}$$

See Table 6.1 for the contribution to ϵ_3 in Minimal SUSY $SU(5)$, and in an $SU(5)$ and $SO(10)$ model with natural Higgs doublet-triplet splitting.

Recent Super-Kamiokande bounds on the proton lifetime severely constrain these dimension 6 and 5 operators with $\tau_{(p \to e^+ \pi^0)} > 1.2 \times 10^{34}$ years and $\tau_{(p \to K^+ \bar{\nu})} > 3.9 \times 10^{33}$ years (205.7 ktyr) at (90% CL) based on the listed exposures [101]. These constraints are now sufficient to rule out minimal SUSY $SU(5)$ [102, 103]. The upper bound on the proton lifetime from these theories (particularly from dimension 5 operators) is approximately a factor of 5 above the experimental bounds. These theories are also being pushed to their theoretical limits. Hence if four dimensional SUSY GUTs are correct, the observation of nucleon decay may be within reach. Two new detectors, DUNE in the U.S. and Hyper-Kamiokande in Japan, are being proposed to search for nucleon decay with a sensitivity to lifetimes of order 10^{35} years after 10 years of running. The discovery of nucleon decay would be a huge success for grand unified theories.

Table 6.1 Contribution to ϵ_3 in three different GUT models

Model	Minimal SU_5	SU_5 "Natural" D/T [97]	Minimal SO_{10} [98, 99]
$\epsilon_3^{\text{GUT breaking}}$	0	-7.7%	-10%
$\epsilon_3^{\text{Higgs}}$	-4%	$+3.7\%$	$+6\%$
M_T^{eff} [GeV]	2×10^{14}	3×10^{18}	6×10^{19}

Chapter 7
SUSY Flavor Problem

Up until now we have discussed the MSSM and SUSY GUTs which define the boundary conditions for the renormalization group equations at the GUT scale. The direct consequence of 4D SUSY GUTs is gauge coupling unification which requires low energy SUSY breaking and a concomitant solution to the gauge hierarchy problem. In addition observations made at the LHC will provide direct information about those GUT boundary conditions, due to the assumed SUSY desert between M_Z and M_G. Finally nucleon decay is predicted at a rate which may be observable at future proton decay experiments, such as Hyper-Kamiokande in Japan or DUNE in the U.S.

But SUSY GUTs provide a framework for much more. In particular, it can minimize the number of fundamental parameters in the theory. It already relates quarks and leptons by placing them in irreducible representations of the GUT group. Thus relating quark and lepton masses. If we now introduce family symmetries we can, in principle, reduce the number of fundamental parameters even further. This will have the effect of correlating much low energy data and thus provide non-trivial tests of the theory.

Before we discuss the general issue of flavor problems in the MSSM, let's consider the well-studied scenario known as minimal flavor violation [MFV]. In this scenario [defined at the GUT scale] we take the GUT boundary conditions for the soft SUSY breaking parameters to be those given in Eqs. (4.16) and (4.17). Given these boundary conditions defined at the GUT scale, one can readily see that the only flavor violation entering into the low energy theory must necessarily be proportional to the CKM matrix. According to Eq. (2.26) the fermion Yukawa matrices are diagonalized via the following redefinitions of the fermion fields:

$$\begin{aligned} u_0 &\equiv U_q\, u, \quad \bar{u}_0 \equiv \bar{u}\, U_{\bar{u}}^{\dagger} \quad d_0 \equiv U_d\, d, \quad \bar{d}_0 \equiv \bar{d}\, U_{\bar{d}}^{\dagger} \\ e_0 &\equiv U_l\, e, \quad \bar{e}_0 \equiv \bar{e}\, U_{\bar{e}}^{\dagger} \quad \nu_0 \equiv U_l\, \nu, \quad \bar{\nu}_0 \equiv \bar{\nu}\, U_{\bar{\nu}}^{\dagger} \end{aligned} \tag{7.1}$$

and the CKM matrix, $V_{CKM} = U_q^{\dagger}\, U_d$, while neutrinos are in the flavor basis.

S. Raby, *Supersymmetric Grand Unified Theories*, Lecture Notes in Physics 939,
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Note these redefinitions of the fermion fields were defined in the SM. Now for the MSSM we shall define what is called the SUSY flavor basis, i.e. we rotate superfields by the same transformations as the fermions. In this SUSY flavor basis, soft SUSY breaking matrices are typically not diagonal, except in case of minimal flavor violation. Clearly the terms in the superpotential, Eq. (3.13), are diagonalized by the transformations of Eq. (7.1). Now consider the soft SUSY breaking Lagrangian, Eq. (4.15). Since in the MFV basis, all soft scalar masses are proportional to the identity matrix at the GUT scale, they remain proportional to the identity matrix after the change of basis. In addition the A terms in the MFV basis are proportional to Yukawa matrices so they are also diagonalized. Hence the only non-diagonal terms in the SUSY Lagrangian at the GUT scale are in the couplings of the $W^{\pm}$ gauge sector to quarks and the non-diagonalization of the PMNS matrix. The latter is negligible and the former is proportional to V_{CKM}. Thus we have shown that flavor violation is minimized with MFV boundary conditions at the GUT scale. Of course, once we use the RG equations below the GUT scale, scalar masses and A terms are no longer diagonal. They contribute new flavor violating processes at the weak scale.

Let us now consider limits on flavor violation coming from low energy data. We shall see that the fundamental theory must necessarily be close to the MFV limit. Once we have motivated MFV phenomenologically, we will then consider soft SUSY breaking mechanisms which can lead to MFV. Consider generic soft terms, for example in the electron–muon system,

$$-\mathscr{L}_{soft} \supset (\tilde{e}^* \quad \tilde{\mu}^*) \begin{pmatrix} m_{11}^2 & m_{12}^2 \\ m_{21}^2 & m_{22}^2 \end{pmatrix} \begin{pmatrix} \tilde{e} \\ \tilde{\mu} \end{pmatrix}. \tag{7.2}$$

Define $\bar{m}^2 = \frac{m_{11}^2 + m_{22}^2}{2}$ and $\delta_{e\mu}^{LL} = \frac{m_{12}^2}{\bar{m}^2}$ and assume $\delta_{e\mu}^{LL} \ll 1$. Using the approximation of small $\delta_{e\mu}^{LL}$ one can calculate the process $\mu \to e\,\gamma$ (Fig. 7.1) and one obtains the following bound [104]

$$|\delta_{e\mu}^{LL}| \leq 4.4 \times 10^{-4} \left(\frac{\tilde{m}}{1\,\text{TeV}}\right)^2 \left(\frac{50}{\tan\beta}\right) \left(\frac{BR(\mu \to e\,\gamma)}{5.7 \times 10^{-13}}\right)^{1/2} \tag{7.3}$$

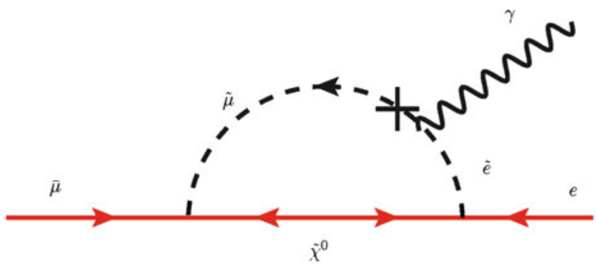

Fig. 7.1 The slepton-neutralino loop contributing to the branching ratio, $BR(\mu \to e\,\gamma)$

Table 7.1 Experimental bounds on charged lepton flavor violating processes

Branching ratio	Experimental bound	Experiment	References
$\mathrm{BR}(\mu \to e\gamma)$	5.7×10^{-13}	MEG	[105]
$\mathrm{BR}(\mu^+ \to e^+ e^+ e^-)$	1.0×10^{-12}	SUNDRUM	[106]
$\mathrm{BR}(\mu^- Au \to e^- Au)$	7.0×10^{-13}	SUNDRUM II	[107]
$\mathrm{BR}(\tau \to \mu\gamma)$	4.5×10^{-8}	Belle	[108]

Table 7.2 Future expected sensitivities on charged lepton flavor violating processes

Branching ratio	Experimental bound	Experiment
$\mathrm{BR}(\mu \to e\gamma)$	$< 6 \times 10^{-14}$	MEG [109]
$\mathrm{BR}(\mu^+ \to e^+ e^+ e^-)$	$\leq 10^{-16}$	Mu3e [110]
$\mathrm{BR}(\mu^- N \to e^- N)$	6.0×10^{-17}	Mu2e, COMET

Table 7.3 Experimental bounds on charged lepton electric dipole moments

Electric dipole moments	Experimental bound	References
d_e	$< 8.7 \times 10^{-29}$ ecm	[111]
d_μ	$= -0.1 \pm 0.9 \times 10^{-19}$ ecm	[47]

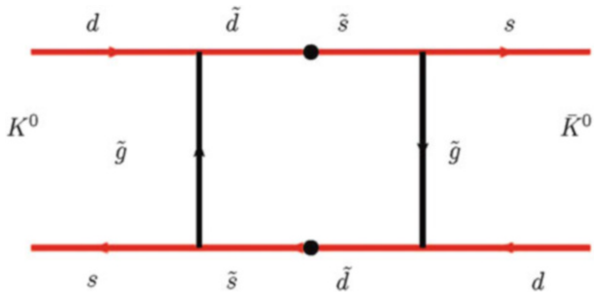

Fig. 7.2 $K^0 - \bar{K}^0$ mixing via gluino exchange

using the present MEG bound on the branching ratio, $BR(\mu \to e\,\gamma) < 5.7 \times 10^{-13}$ at 90% CL [105].

The experimental bounds on lepton flavor violating processes are quite severe. We have some experimental bounds in Table 7.1. In the future, the sensitivities are expected to be even better with some examples in Table 7.2. There are also stringent limits on electric dipole moments of charged leptons, Table 7.3.

Flavor violation in the quark sector is also very restricted, especially for the first two families. The strongest bound comes from $K^0 - \bar{K}^0$ mixing and the CP violating parameter, ϵ_K, (see Fig. 7.2).

We have the bounds,

$$\Delta m_K \Rightarrow \sqrt{|Re\delta_{ds}^{LL}|^2} \leq 0.04\ (0.08) \left(\frac{m_{\tilde{q}}}{1\ \text{TeV}}\right) \tag{7.4}$$

$$\epsilon_K \Rightarrow \sqrt{|Im\delta_{ds}^{LL}|^2} \leq 3.0\ (6.4) \times 10^{-3} \left(\frac{m_{\tilde{q}}}{1\ \text{TeV}}\right) \tag{7.5}$$

for $\frac{m_{\tilde{g}}^2}{m_{\tilde{q}}^2} = 0.3\ (1.0)$ [104].

Let us consider some possible solutions to the general SUSY flavor and CP problems. The problem is that squark and slepton mass matrices are, in general, not diagonal in the SUSY flavor basis, i.e. in the same basis that quark and lepton mass matrices are diagonal. For example, consider for just the first two families of quarks, in the quark mass basis,

$$m_Q = \begin{pmatrix} m_d & 0 \\ 0 & m_s \end{pmatrix} \tag{7.6}$$

$$m_{\tilde{Q}}^2 = \begin{pmatrix} c_L & -s_L \\ s_L & c_L \end{pmatrix} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \begin{pmatrix} c_L & s_L \\ -s_L & c_L \end{pmatrix}$$

where the matrices containing $c_L = \cos\theta_L,\ s_L = \sin\theta_L$ are the additional matrices needed to diagonalize the squark mass squared matrix. Multiplying the matrices we have

$$m_{\tilde{Q}}^2 = \frac{(m_1^2 + m_2^2)}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{(m_1^2 - m_2^2)}{2} \begin{pmatrix} \cos 2\theta_L & \sin 2\theta_L \\ \sin 2\theta_L & -\cos 2\theta_L \end{pmatrix}. \tag{7.7}$$

Thus the constrained parameter, δ_{ds}^{LL}, is given by

$$\delta_{ds}^{LL} = \frac{\frac{(m_1^2 - m_2^2)}{2} \sin 2\theta_L}{\bar{m}^2} \ll 1 \tag{7.8}$$

where $\bar{m}^2 = \frac{(m_1^2 + m_2^2)}{2}$. Note, that the same problem exists for δ^{RR} and δ^{LR}, since they all appear in the scalar mass squared matrix.

There are, in general, three possible solutions for the appropriate $\delta \ll 1$. They are

1. Universal scalar masses for states with the same SM charge. In the simple example, $m_1^2 = m_2^2 = m_0^2$ and also $A_{ij} = A_0 1_{2\times 2}$;
2. $\sin 2\theta_L \ll 1$, which corresponds to aligning the scalar mass matrix with the fermion mass matrix, and
3. $\bar{m}^2 \gg \text{TeV}^2$ for the first two families of scalars, since the most stringent constraints affect the first two families. This is known as decoupling.

7.1 SUSY Breaking Mechanisms

So what SUSY breaking mechanisms satisfy one or more of these constraints? Earlier we discussed Minimal Flavor Violation as satisfied by the GUT boundary conditions of the CMSSM in Eqs. (4.16) and (4.17). This is an example of solution 1 (universal scalar masses).

Solution 2 (alignment) can be obtained by invoking non-abelian flavor symmetries. When the flavor symmetries between different families are exact then the scalar masses are universal, i.e. proportional to the unit matrix. Then the fermion mass matrices are only generated when the family symmetries are spontaneously broken. In the process, generating the hierarchy of fermion masses. In addition, the off diagonal elements of the scalar mass squared matrices are also only generated when the family symmetries are spontaneously broken. In this case the scalar mass squared matrices are aligned with their fermionic partners.

Finally splitting the third family masses from the first two would be permitted if the family symmetry was $SU(2)$ (or a discrete subgroup) where the first two families are in a doublet and the third family and the Higgses are in a singlet under the family symmetry. Then the universal first two family scalar masses can be made much larger than the third family scalars (hence the first two family scalars decouple).

But do we know of any natural SUSY breaking mechanism which gives any of the solutions? In Sect. 4.1 we mentioned the two SUSY breaking transmission mechanisms available in four space-time dimensional SUSY theories, i.e. gauge-mediated, gravity-mediated SUSY breaking and the SUSY breaking mechanism, gaugino condensates. In all cases, SUSY breaking occurs in a hidden sector of the theory and then is transmitted to the visible sector via flavor blind gauge interactions (see Fig. 7.3).

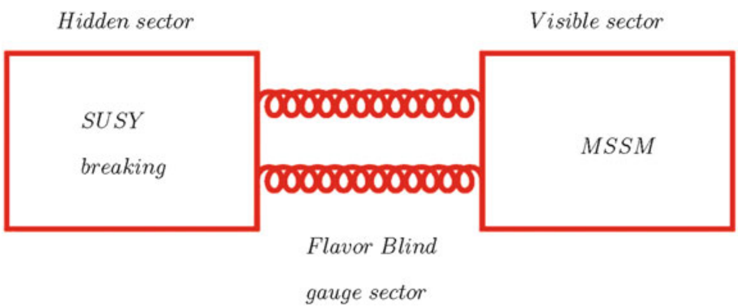

Fig. 7.3 Gauge- (or Gravity-) mediated SUSY breaking has a hidden sector where SUSY breaking occurs, a visible sector which is the MSSM and the mediator which is the Standard Model gauge bosons (or Gravity)

Gauge Mediated SUSY Breaking

Gauge mediated SUSY breaking requires a SUSY breaking sector which spontaneously generates F-term SUSY breaking. Consider, for example, a superfield X which obtains a VEV given by

$$\langle X(x_\mu, \theta_\alpha)\rangle = X + (\theta)^2\, F_X. \tag{7.9}$$

The field X does not couple directly to the visible sector (quarks, leptons and gauge bosons). Instead it couples to messenger fields, M, which have Standard Model gauge quantum numbers. Let's take two superfields, $M,\ \bar{M} \subset \mathbf{5},\ \bar{\mathbf{5}}$ of $SU(5)$ and add a term to the superpotential of the form $\mathscr{W} \supset \lambda\, X\, \bar{M}\, M$ plus the terms necessary to give X the appropriate VEV. Then gauginos obtain a SUSY breaking mass via the one loop corrections, Fig. 7.4. The gauginos obtain mass given by

$$M_i \approx \frac{\alpha_i}{4\pi}\, \Lambda_{eff}, \quad \text{with} \quad \Lambda_{eff} = \frac{F_X}{X}. \tag{7.10}$$

Scalars obtain mass at two loops via the Feynman diagram, Fig. 7.5. We find

$$\tilde{m}_\alpha^2 = 2\sum_{i=1}^{3} C_\alpha^i (\frac{\alpha_i}{4\pi}\, \Lambda_{eff})^2 \tag{7.11}$$

where C_α^i are the quadratic Casimirs for scalars in the representation α under the SM gauge group, i.e. $C^3_{\text{color triplet}} = 4/3,\ \ C^2_{\text{weak doublet}} = 3/4,\ \ C^1_{Y/2} = \frac{3}{5}\,(Y/2)^2$.

Note, SUSY is spontaneously broken at the scale $\sqrt{F_X}$. However, the effective SUSY breaking scale, in the SM sector [Eq. (7.10)], is suppressed by the messenger scale, X. Keeping F_X fixed and increasing X we see that SUSY breaking decouples from the SM sector [23, 112], since it must necessarily be transmitted to the visible sector by heavy messengers.

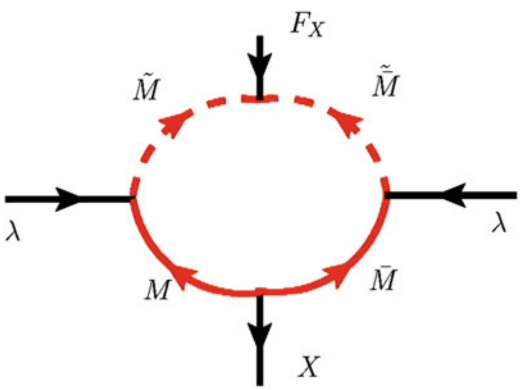

Fig. 7.4 The messenger fields enter the loop to give mass to gauginos at the messenger scale, $m_M = \lambda\, X$

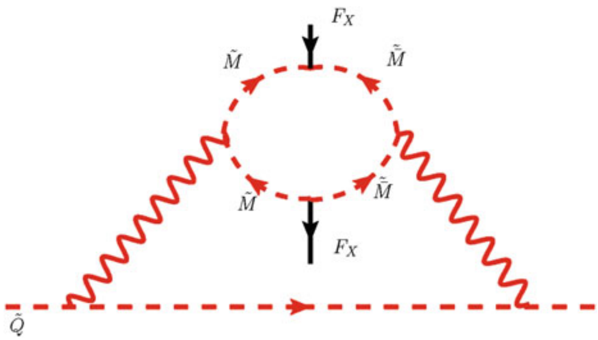

Fig. 7.5 The messenger fields enter the loop to give mass to scalars at two loops

Gravity Mediated SUSY Breaking

In gravity mediated SUSY breaking, the mediator is gravity itself. In supergravity [SUGRA] models with a flat Kähler potential the gravitino mass is given by

$$m_{3/2} \sim \frac{F_X}{\sqrt{3}\, m_{Pl}}. \tag{7.12}$$

where m_{Pl} is the reduced Planck scale, i.e. $m_{Pl} = M_{Pl}/\sqrt{8\pi}$. As we see in Eq. (7.12), the gravitino mass, and then all SUSY breaking parameters in the visible sector, are suppressed by m_{Pl}. Thus, again, keeping F_X fixed and taking m_{Pl} to infinity, soft SUSY breaking effects decouple.

The simplest SUGRA model is known as the Polonyi model [113, 114]. In this model the superpotential is given by

$$\mathscr{W} = m\, m_{Pl}(X + \beta) + \mathscr{W}_{MSSM}(\Phi_i) \tag{7.13}$$

[where $\mathscr{W}_{MSSM}$ is the superpotential for the MSSM given in Eq. (3.13)], a flat Kähler potential

$$\mathscr{K} = X^* X + \mathscr{K}_{MSSM} \tag{7.14}$$

[where $\mathscr{K}_{MSSM}$ is given in Eq. (3.12)] and the gauge kinetic function, $f(\Phi)$, given in Eq. (3.18) is taken to be $f(X) = X/\hat{M}$ where $\hat{M} \approx m_{Pl}$. Minimizing the scalar potential with respect to the scalar field, X, and requiring zero cosmological constant one finds

$$X = (\sqrt{3} \mp 1)\, m_{Pl} \tag{7.15}$$

with β fine-tuned to take the value $\beta = (\pm 2 - \sqrt{3})\ m_{Pl}$. Supersymmetry is spontaneously broken since $F_X = m\ m_{Pl} \neq 0$ and the gravitino obtains mass $m_{3/2} = m\ e^{(2 \mp \sqrt{3})}$.

In the limit $m_{Pl} \to \infty$ with $m_{3/2}$ fixed, we find the scalar potential for MSSM fields, $\{\Phi_i\}$, given by [see Eq. (4.11)]

$$V(\langle X \rangle, \Phi_i) = e^{(|\langle X \rangle|^2/m_{Pl}^2)} \left[\sum_i |\frac{\partial \mathscr{W}}{\partial \phi_i}|^2 + m_{3/2}^2 \sum_i |\phi_i|^2 \right.$$
$$\left. + m_{3/2} \left(\sum_i \phi_i \frac{\partial \mathscr{W}}{\partial \phi_i} + (A - 3)\ \mathscr{W} + h.c. \right) \right] \tag{7.16}$$

with $A = 3 \mp \sqrt{3}$ [115, 116]. In this simplest of scenarios with flat Kähler potential and minimal Polonyi superpotential, scalars obtain a universal scalar mass, $m_{3/2}$, a universal A term, $A_0 = A\ m_{3/2}$, a universal B term, $B = (A - 1)\ m_{3/2}$ and universal gaugino masses, $M_{1/2} = F_X/m_{Pl} \sim m_{3/2}$.

Unfortunately, none of this is guaranteed by any symmetry of the theory. In principle, one could add terms to the Kähler potential of the form

$$\mathscr{K} \supset \frac{X^*\ X}{m_{Pl}^2}\ \kappa_{ij}\ \Phi_i^*\ e^{-2\ g\ V}\ \Phi_j \tag{7.17}$$

which couples the Polonyi field, X, to the matter fields, Φ_i, and where κ_{ij} is an arbitrary block diagonal matrix coupling fields with the same SM charges. As a consequence scalar masses would obtain non-universal corrections,

$$\Delta \tilde{m}_{ij}^2 \propto \kappa_{ij}\ m_{3/2}^2. \tag{7.18}$$

Similarly if one coupled the Polonyi field directly to matter fields in the superpotential then A terms can also obtain non-universal corrections. These could be disastrous phenomenologically. Finally, gaugino masses may not be universal if the gauge kinetic term was of the form

$$\mathscr{L}_{gauge-kinetic} = [\frac{1}{8g_s^2} \int d^2\theta Tr(\frac{X}{M_3}\ \mathscr{W}_g{}^\alpha\ \mathscr{W}_{g\alpha}) + h.c.] \tag{7.19}$$
$$+ [\frac{1}{8g^2} \int d^2\theta Tr(\frac{X}{M_2}\ \mathscr{W}_W{}^\alpha\ \mathscr{W}_{W\alpha}) + h.c.]$$
$$+ [\frac{1}{16g'^2} \int d^2\theta (\frac{X}{M_1}\ \mathscr{W}_b{}^\alpha\ \mathscr{W}_{b\alpha}) + h.c.]$$

with $M_i, i = 1, 2, 3$ all different. Of course, if the theory had an $SU(5)$ symmetry, then symmetry arguments would require $M_3 = M_2 = M_1 \equiv M$ at M_G. However, there is, in general, no symmetry in the scalar sector, UNLESS we introduce non-abelian flavor symmetries (sometimes called family or horizontal symmetries) which relate matter fields with the same SM charges. Only then can we have the possibility of guaranteeing universal scalar masses, at least until those same family symmetries are broken. But why do they have to be broken you ask? Because the same family symmetries would constrain the fermion mass matrices. We will discuss this in more detail later in the course when we discuss particular SUSY GUT models.

μ Problem

We can now discuss the μ problem, i.e. why is $\mu \ll M_{GUT}$ or M_{Pl}. We first require that the μ term vanishes at tree level in the SUSY Lagrangian. Then one possible way of generating the μ term is via the Giudice-Masiero mechanism [117]. In this case we add a term to the Kähler potential of the form

$$\Delta\mathscr{K} = \frac{X^*}{M_{Pl}} H_u H_d. \tag{7.20}$$

Thus when F_X gets a SUSY breaking VEV we obtain $\mu = \frac{F_X}{M_{Pl}} \sim m_{3/2}$. We can also add terms to the super potential of the form

$$\Delta\mathscr{W} = \frac{\Phi^2}{M_{Pl}} H_u H_d. \tag{7.21}$$

If Φ obtains a non-zero SUSY conserving VEV of the form $\langle\Phi\rangle = M$, then the low energy theory will contain a μ term given by $\mu = M^2/M_{Pl}$. Note, this theory typically has a global $U(1)$ symmetry. When Φ obtains a non-zero VEV a Goldstone boson is created, i.e. the axion which may solve the strong CP problem. With $M \sim 10^{10}$ GeV we find $\mu \sim 100$ GeV. This is the Kim-Nilles mechanism [118].

Dynamical SUSY Breaking in Supergravity

It was shown in [28] that gaugino condensation can result in dynamical SUSY breaking. The pure supergravity Lagrangian is given by

$$\begin{aligned}\mathscr{L}_{SG} = &-\frac{1}{6} e f(\tilde{s},\ \tilde{s}^*)\ [R(e,\ \omega(e,\psi)) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\ \gamma_5\gamma_\nu\ D_\rho(\omega(e,\psi))\ \psi_\sigma \\ &-\frac{1}{3} e u u^\dagger + \frac{1}{3} e A_m A^m]\end{aligned} \tag{7.22}$$

where e is the determinant of the vierbein, ω is the Lorentz gauge function, ψ_μ is the gravitino field and $u,\ A_m$ are gravity auxiliary fields. Using the supergravity formalism in [28, 119] one has the supergravity Lagrangian coupling a chiral superfield S to the non-abelian gauge field V via

$$\mathscr{L} = \int d^4x d^4\theta\ E\,(f(S,\ S^*) + Re\frac{1}{R}\mathscr{W} + Re\frac{1}{R}f(S)\mathscr{W}^\alpha \mathscr{W}_\alpha) \tag{7.23}$$

where $f(S,\ S^*) = -3m_{Pl}^2 e^{-\mathscr{K}/3m_{Pl}^2}$, $f(S)$ is the gauge kinetic function, E is the superspace determinant and R is the chiral scalar curvature superfield [120]. For canonical Kähler potential and assuming, due to strong gauge interactions, a gaugino condensate

$$\langle \lambda^\alpha\ \lambda_\alpha \rangle \sim \mu^3 \tag{7.24}$$

forms, then SUSY is spontaneously broken. Moreover

$$F_S = \frac{1}{4} f'(S) \langle \lambda^\alpha\ \lambda_\alpha \rangle \tag{7.25}$$

giving a gravitino mass

$$m_{3/2} = \frac{\mu^3}{4\sqrt{3}\ m_{Pl}^2}. \tag{7.26}$$

Summary

We conclude this discussion of SUSY flavor and CP problems with a brief summary. Experimental evidence shows that the fermion and scalar mass matrices are necessarily *almost* diagonal in the SUSY flavor basis. Off-diagonal elements in the scalar mass matrices must be small. This is particularly true for the first two families from processes such as $\mu \to e\ \gamma$, $K^0 - \bar{K}^0$ mixing or $K_L \to \mu\ \bar{e}$. The imaginary part of the diagonal elements are also constrained by the electric dipole moments of the electron and neutron and the imaginary part of the off-diagonal element is constrained by CP violation in the K system, i.e. ϵ_K. Thus SUSY breaking mechanisms which naturally provide minimal flavor violation are necessary. We discussed gauge-mediated SUSY breaking which satisfies this criteria. Gravity-mediated SUSY breaking, on the other hand, can only satisfy this criteria with the additional imposition of flavor symmetries.

Chapter 8
SUSY GUTs and the CP and Flavor Problem

In Sect. 6.4 we discussed two experimental consequences of SUSY GUTs, i.e. gauge coupling unification and proton decay. It might be expected that since the GUT scale is so high, these might be the only observable consequences at low energies. The latter simply because the effective low energy theory conserves baryon number and only weakly violates lepton number. In this section we argue that radiative corrections at the GUT scale can also affect low energy flavor physics. The reason is that these radiative corrections can induce flavor violation in scalar masses and then these scalar masses enter quantum loops at the weak scale (see [121] and [122]).

Consider extending the MSSM by terms in the superpotential of the form [121]

$$\mathscr{W} = F_i \, \eta_{ij} \, F_j \, X + F_i \, \xi_i \, Y \, Z \tag{8.1}$$

where F_i is one member of the set $F_i = \{Q_i, \ \bar{U}_i, \ \bar{D}_i, \ L_i, \ \bar{E}_i, \ \bar{N}_i\}$ and $X, \ Y, \ Z$ are superfields with the correct gauge quantum numbers to couple to quarks and leptons. These terms in the superpotential induce radiative corrections to squark or slepton masses (via the one loop graphs of Fig. 8.1) of the form

$$\Delta m^2_{\tilde{F}\, ij} \propto (\eta^\dagger_{ik} \, \eta_{kj} + \xi^\dagger_i \, \xi_j) \ \log(\frac{M_{Pl}}{M_{(X, \ Y, \ Z)}}). \tag{8.2}$$

These corrections to the scalar mass squared matrices can, in principle, violate flavor symmetries in the low energy theory. Hence SUSY theories are sensitive to physics at high energies! In the next sections we consider two well studied cases.

S. Raby, *Supersymmetric Grand Unified Theories*, Lecture Notes in Physics 939,
DOI 10.1007/978-3-319-55255-2_8

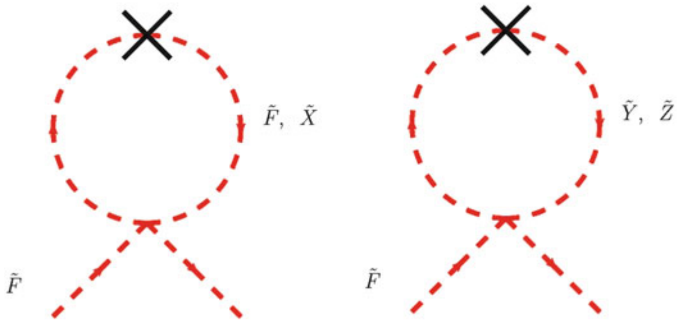

Fig. 8.1 One loop corrections to scalar quark or lepton masses

8.1 Radiative Corrections Due to Physics Above the GUT Scale

Once SUSY is broken we obtain soft breaking terms which can affect flavor violation. These are, in particular, the mass squared terms of the form $\tilde{f}^\dagger\, m_f^2\, \tilde{f}$ or A terms of the form $\tilde{f}\, A_f\, \tilde{f}\, h$ where the scalar fields $\tilde{f}$ and h are any one in the set $\tilde{f} = \{\tilde{q},\ \tilde{\bar{u}},\ \tilde{\bar{d}},\ \tilde{l},\ \tilde{\bar{e}},\ \tilde{\bar{\nu}}\}$ and $h = \{h_u,\ h_d\}$ in gauge invariant combinations [see Eq. (4.15)]. In Eqs. (4.16) and (4.17) we chose minimal flavor violating boundary conditions at the GUT scale given by

- scalar mass—

$$m_f^2|_{M_G} = m_0^2\ 1_{3\times 3}; \tag{8.3}$$

- A parameter—

$$A_f^{ij} = A_0\ Y_f^{ij}. \tag{8.4}$$

However if we have gravity mediated SUSY breaking then the MFV boundary conditions are properly imposed at the fundamental scale $m_{Pl} \approx 2.4 \times 10^{18}$ GeV and not $M_G \approx 2 \times 10^{16}$ GeV. In gauge mediated SUSY breaking, on the other hand, the boundary conditions are determined at the messenger scale which may be below the GUT scale. We now show that additional flavor violating effects are induced in gravity mediated SUSY breaking due to radiative corrections above the GUT scale.

Consider supergravity with MFV boundary conditions imposed at the Planck scale

- scalar mass—

$$m_f^2|_{M_{Pl}} = m_0^2\ 1_{3\times 3}; \tag{8.5}$$

- *A* parameter—

$$A_f^{ij} = A_0 \, Y_f^{ij}. \tag{8.6}$$

In $SU(5)$, the superpotential has terms of the form in Eq. (6.4). At the GUT scale we split the Higgs doublets and triplets such that below the GUT scale we only have the states in the MSSM, i.e. the Higgs triplets obtain mass at the GUT scale. The superpotential above the GUT scale contains not only the interactions of the Higgs doublets,

$$\mathscr{W} \supset (Y_u)^{ij} \, \bar{U}_i \, Q_j \, H_u + (Y_d)^{ij} \, (\bar{D}_j \, Q_i + \bar{E}_i \, L'_j) \, H'_d + (Y_\nu)^{ij} \, \bar{N}_i \, H_u \, L'_j - \frac{1}{2} M_{ij} \, \bar{N}_i \, \bar{N}_j. \tag{8.7}$$

[see Eq. (5.46)], but also the interactions of the Higgs triplets, $T, \ \bar{T}$,

$$\mathscr{W} \supset \ (Y_u)^{ij} \, (\bar{U}_i \, \bar{E}_j + Q_i \, Q_j) \, T + (Y_d)^{ij} \, \bar{T} \, (\bar{U}_i \, \bar{D}_j + Q_i \, L'_j) + (Y_\nu)^{ij} \, \bar{N}_i \, T \, \bar{D}_j. \tag{8.8}$$

These terms will induce flavor violation in both the quark and lepton sectors. The analysis is typically performed in the SUSY flavor basis [121] where the up quark Yukawa matrix is diagonalized via unitary rotations on up quark superfields. In this basis neither the down quark nor the lepton Yukawa matrices are diagonal.

For example, in $SU(5)$ if we renormalize the slepton masses from M_{Pl} to M_G, taking into account the large top quark Yukawa coupling in the loop (left, Fig. 8.2) we obtain flavor violating mass squared terms due to the interaction

$$\mathscr{W} \supset \bar{U} \, (Y_u^D \, V_{CKM}) \, \bar{E} \, T. \tag{8.9}$$

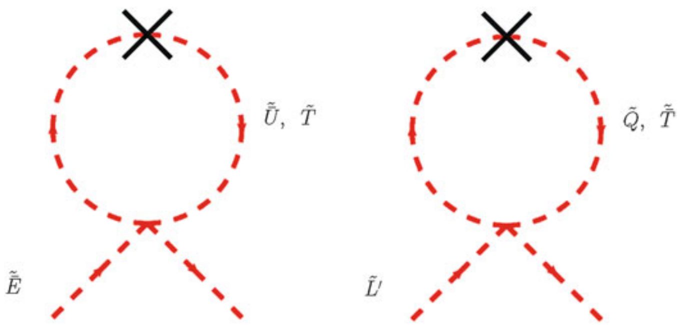

Fig. 8.2 One loop correction to slepton mass due to RG running from M_{Pl} to M_G. In the case of $SU(5)$ only the contribution on the left is significant and proportional to the top quark Yukawa coupling. However, for $SO(10)$ with Yukawa unification, both terms are important

It is of the form

$$\Delta m^2_{\tilde{e}\,ij} = -V^{\dagger}_{CKM\,i3}\, V_{CKM\,3j}\, I \tag{8.10}$$

with $I = \frac{3}{8\pi^2} \int_{M_G}^{M_{Pl}} Y_{33}^2\, (m^2_{\tilde{u}} + 2m^2_{\bar{T}} + A^2_{33}) d(\ln \mu)$. This is an approximation assuming that $Y_{33} \approx Y_t$, $A_{33} \approx A_t$ are dominated by the top quark Yukawa coupling [123, 124]. Note, flavor violating effects are induced in the lepton sector due to the mixing of quarks and leptons above the GUT scale. A similar flavor violating contribution is induced in the leptonic matrix, A_e. For $SO(10)$ with Yukawa unification, both graphs in Fig. 8.2 give large contributions to $\Delta m^2_{\tilde{\bar{e}}\,ij}$ and $\Delta m^2_{\tilde{e}\,ij}$. A detailed analysis of these flavor violating effects to processes like $\mu \to e\,\gamma$, $\mu\,N \to e\,N$, $\tau \to \mu\,\gamma$ for both $SU(5)$ and $SO(10)$ were carried out in [124] assuming the minimal theory above M_G. The resulting contribution to (for example) $\mu \to e\,\gamma$ is largest for $SO(10)$ and scales as $\tilde{m}^{-4}\,\frac{\Delta\tilde{m}^2}{\tilde{m}^2}$ and decreases with Y_t. From Fig. 12, [124] one concludes that with scalar masses of order 3 TeV, $\mu \sim$ TeV and $\tan\beta \sim 50$, the branching ratio for $\mu \to e\,\gamma$ is most likely below the present experimental bounds. On the other hand, this analysis assumed a minimal theory above M_G. However given the hierarchy of fermion masses and possible Froggatt-Nielsen [125] mechanism for this hierarchy, it is not at all clear that these flavor violating effects are consistent with the data. In fact they typically induce much larger flavor violating effects. As far as I am aware, a detailed analysis of such theories has not been performed.

In addition to flavor violating effects, CP violating effects from physics above the GUT scale are also induced. See for example [124, 126]. They find [124]

$$\frac{|d_e|}{10^{-27}\,\mathrm{ecm}} = 0.74 \sin\phi\, \sqrt{\frac{BR(\mu \to e\,\gamma)}{5.7 \times 10^{-13}}} \tag{8.11}$$

where ϕ is the CP violating phase and the present experimental bounds [47] are $BR(\mu \to e\,\gamma) < 5.7 \times 10^{-13}$ and $d_e < 10.5 \times 10^{-28}$ ecm. Again, the results are likely consistent with the data.

8.2 Radiative Corrections Due to Right-Handed Neutrinos

We know that neutrinos have mass and, within the context of GUTs, the most natural scenario is the See-Saw mechanism. With right-handed neutrino Majorana masses of order $10^{12\pm2}$ GeV, we obtain light active neutrinos with Majorana masses consistent with oscillation data. $SO(10)$ and PS GUTs are more constrained than $SU(5)$ because in $SO(10)$ and PS, Dirac neutrino masses are related to up quark masses by the symmetry. The superpotential below M_G and above the right-handed neutrino masses contains the term [Eq. (6.4)]

$$\begin{aligned} \mathscr{W} &\supset (Y_\nu)^{ij}\, \bar{N}_i\, H_u\, L'_j - \tfrac{1}{2} M_{ij}\, \bar{N}_i\, \bar{N}_j \\ &= \bar{N}^T\, Y_\nu\, H_u\, L' - \tfrac{1}{2}\, \bar{N}^T\, M\, \bar{N}. \end{aligned} \tag{8.12}$$

The heavy right-handed neutrinos, $\bar{N}_i$, can be integrated out of the theory in a supersymmetric way by solving the equations,

$$\frac{\partial \mathscr{W}}{\partial \bar{N}_i} = (Y_\nu)^{ij}\, H_u\, L'_j - M_{ij}\, \bar{N}_j = 0 \tag{8.13}$$

and plugging the solution for $\bar{N}_i$ back into the superpotential.[1] We find

$$\bar{N} = M^{-1}\, Y_\nu\, H_u\, L' \tag{8.14}$$

and thus we obtain the supersymmetric version of the Weinberg operator,

$$\mathscr{W} \supset \frac{1}{2} (H_u\, L')^T\, Y_\nu^T\, M^{-1}\, Y_\nu\, (H_u\, L'). \tag{8.15}$$

When electroweak symmetry is broken and $\langle H_u \rangle = \begin{pmatrix} 0 \\ v_u \end{pmatrix}$ we find the Majorana neutrino mass term for the light active neutrinos given by

$$-\mathscr{L}_\nu = \frac{1}{2} \nu^T\, m_\nu^T\, M^{-1}\, m_\nu\, \nu \tag{8.16}$$

with $m_\nu = Y_\nu\, v_u$. In the neutrino flavor basis we then obtain the result in Eq. (2.29).

If we include the right-handed neutrinos in the renormalization group equations from M_G (or M_{Pl}) to M_N we obtain radiative corrections to slepton mass squared matrices given by Borzumati and Masiero [122], Gabbiani and Masiero [127], Calibbi et al. [128]

$$(\Delta m^2_{\tilde{e}})_{ij} = -\frac{3m_0^2 + A_0^2}{16\pi^2} \sum_k Y_{\nu ik}\, Y^\dagger_{\nu kj}\, ln(\frac{M_G^2}{M_{Nk}^2}). \tag{8.17}$$

These induce lepton flavor violating interactions in the low energy theory (Fig. 8.3).

[1] This determines an approximate solution to diagonalizing the neutrino mass matrix.

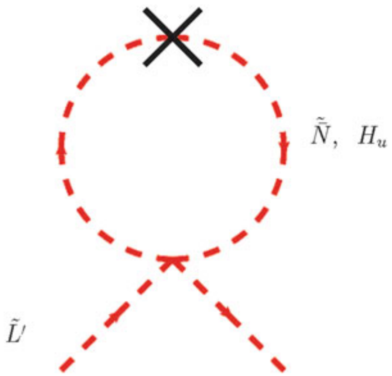

Fig. 8.3 One loop correction to slepton mass due to RG running from M_G to M_{M_N}

8.3 Summary

Calibbi et al. [128] studies the contribution to lepton flavor violating processes from both physics above the GUT scale and above the Majorana neutrino mass scale. They compare the reach of LFV experimental searches to the LHC reach for supersymmetric particles. They have considered an $SO(10)$ SUSY GUT with $SO(10)$ breaking to $SU(5)$ at a scale $M_X = 5 \times 10^{17}$ GeV which then breaks to the SM at the GUT scale $M_G = 2 \times 10^{16}$ GeV.

For neutrino oscillations they make the following two assumptions for the origin of the large U_{PMNS} mixing angles. In the first case they assume $Y_\nu = Y_u$ (called the CKM case with small mixing angles) and in the second case $Y_\nu = U_{PMNS}(M_{R3})\, Y_u^{diag}$ (called the PMNS case with large mixing angles). Regions of soft SUSY breaking parameter space with light gauginos and scalars are essentially excluded by the present bounds on lepton flavor violation. Even with heavier gauginos and scalars, the second case with large mixing angles is very much constrained. Only the case with small mixing angles has large regions of soft SUSY breaking parameter space (with heavy super partners) which survive. Note, the case with small mixing angles is actually consistent with $SO(10)$ which relates up quark and Dirac neutrino Yukawa couplings.

In [129] the authors compare constraints on GUT physics from both lepton flavor violating reactions and quark flavor violating reactions due to radiative corrections at the GUT scale (above and/or below). Renormalization group running below the Planck scale induces flavor violation in the quark sector due to the term

$$(Y_d)^{ij}\, \bar{T}\, (\bar{U}_i\, \bar{D}_j) \tag{8.18}$$

[Eq. (8.8)] in the superpotential. In $SO(10)$, these terms can be significant and they are severely constrained by quark flavor violating processes, such as $K^0 - \bar{K}^0$, $b \to s\,\gamma$, etc. In this paper the constraints on Yukawa couplings due to GUT relations is used to obtain correlations between constraints on flavor violation between the quark and lepton sectors. For example, leptonic constraints on $\tau \to \mu + \gamma$ places stringent constraints on the mixing ΔM_{B_s}. Clearly these cross correlations will severely constrain any SUSY GUT theory.

Chapter 9
Fermion Masses and Mixing in SUSY GUTs: Predictive Theories

In this section we discuss the construction of complete $SU(5)$ and $SO(10)$ SUSY GUTs. When constructing such a theory one has two goals in mind. In the first place one must choose representations for the new GUT multiplets entering the theory at the GUT scale which are necessary for GUT symmetry breaking and Higgs doublet-triplet splitting. These affect gauge coupling unification and nucleon decay rates. One must also obtain Yukawa couplings which are consistent with low energy physics. In the best case scenario one finds a predictive theory for fermion masses and mixing angles which has fewer parameters at the GUT scale than the Standard Model.

When choosing the new states at the GUT scale one may include large representations of the GUT group, as long as the effective low energy theory below the GUT scale only includes the states and interactions of the MSSM.[1] It is a common approach for both $SU(5)$ and $SO(10)$ models for the Standard Model Higgs bosons to be contained in higher dimensional Higgs representations including for $SU(5)$ the **45** [55] or for $SO(10)$ the $\overline{\mathbf{126}}$ and/or **120** [70, 94, 130–132]. Of course, large dimensional representations offer new problems. Gauge couplings above the GUT scale quickly become non-perturbative and it is non-trivial to obtain a GUT breaking sector which satisfies the requirement that the effective theory below the GUT scale is the MSSM.

It is important to note that grand unification alone is not sufficient to obtain predictive theories of fermion masses and mixing angles. Other ingredients are needed. In one approach additional global family symmetries are introduced (non-abelian family symmetries can significantly reduce the number of arbitrary parameters in

[1]In some cases one may want to include additional low energy states or gauge interactions, but I will not discuss these cases here. For example, the non-minimal supersymmetric standard model [NMSSM] includes an extra SM singlet which might be useful for ameliorating the small hierarchy problem. Also, with the addition of vector-like families it could be made consistent with the recent observed diphoton bump at 750 GeV by ATLAS and CMS.

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DOI 10.1007/978-3-319-55255-2_9

Table 9.1 Patterns of masses and mixing

$\lambda_t = \lambda_b = \lambda_\tau = \lambda_{\nu_\tau}$	$SO(10)@M_G$
$\lambda_s \sim \frac{1}{3}\lambda_\mu, \quad \lambda_d \sim 3\lambda_e$	$@M_G$ [55, 60, 61]
$m_s \approx 4 \cdot \frac{1}{3} m_\mu, \quad m_d \approx 4 \cdot 3 m_e$	$@M_Z$
$\lambda_d \lambda_s \lambda_b \approx \lambda_e \lambda_\mu \lambda_\tau$	$SU(5)@M_G$
$Det(m_d) \approx Det(m_e)$	$@M_G$
$V_{us} \approx (\sqrt{m_d/m_s} - i\sqrt{m_u/m_c})$	[133–136]
$V_{ub}/V_{cb} \approx \sqrt{m_u/m_c}$	[137]
$V_{cb} \sim m_s/m_b \sim \sqrt{m_c/m_t}$	[60, 61]

the Yukawa matrices). These family symmetries constrain the set of effective higher dimensional fermion mass operators. Moreover, sequential breaking of the family symmetry is correlated with the hierarchy of fermion masses. In addition, some simple patterns of fermion masses (see Table 9.1) must be incorporated into any successful model.

Three-family models exist which roughly fit all the data, including neutrino masses and mixing [95–97, 99, 138–147]. We shall only consider two specific examples here, a complete $SU(5)$ example found in [97] and an $SO(10)$ example in [99, 148].

9.1 Complete $SU(5)$ SUSY GUT: GUT Breaking and Doublet-Triplet Splitting

Consider first the complete $SU(5)$ SUSY GUT [97]. The symmetry of the model is SUSY $SU(5) \otimes U(1)$. Define Q as the charge associated with $U(1)$. The superpotential of the model has three parts:

$$\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 \quad . \tag{9.1}$$

The $\mathcal{W}_1$ term only contains the field Y in the **75** representation of $SU(5)$ with $Q = 0$:

$$\mathcal{W}_1 = c_1\, Y^3 + M_Y\, Y^2 \quad . \tag{9.2}$$

The effect of $\mathcal{W}_1$ is to provide Y with a VEV of order $M_Y/c_1 \approx M_Y \approx M_{GUT}$ and to give a mass to all physical components of Y, i.e. those that are not absorbed by the Higgs mechanism (the **75** uniquely breaks $SU(5)$ down to $SU(3) \otimes SU(2) \otimes U(1)$).

The $\mathcal{W}_2$ term induces doublet-triplet splitting:

$$\mathcal{W}_2 = c_2\, H\, Y\, H_{50} \;+\; c_3\, \bar{H}\, Y\, H_{\overline{50}} \;+\; c_4\, H_{50}\, H_{\overline{50}}\, X \quad . \tag{9.3}$$

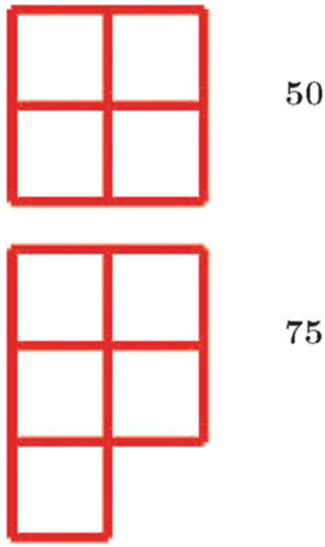

Fig. 9.1 The **75** and **50** dimensional fields in $SU(5)$ are defined in terms of the Young Tableaux

Table 9.2 States and their quantum numbers under $SU(5)$ and Q

Field	Y	H	$\bar{H}$	H_{50}	$H_{\overline{50}}$	X
$SU(5)$	**75**	**5**	$\bar{\mathbf{5}}$	50	$\overline{50}$	**1**
Q	0	-2	1	2	-1	-1

H and $\bar{H}$ are the **5** and $\bar{\mathbf{5}}$ of Higgs fields. Without loss of generality the constants, c_2, c_3 and c_4, real and positive. For a definition of the **75** and **50** dimensional fields in $SU(5)$ see the Young Tableaux, Fig. 9.1. The renormalizable couplings that appear in $\mathscr{W}_2$ are the most general allowed by the $SU(5)$ and the Q assignments of the fields in Eq. (9.3) which are given as follows (Table 9.2).

At the minimum of the potential, in the limit of unbroken SUSY, the VEVs of the fields H, $\bar{H}$, H_{50} and $H_{\overline{50}}$ all vanish, while the X VEV remains undetermined. When SUSY is softly broken the light doublets in H and $\bar{H}$ acquire a small VEV, while the X VEV is fixed near the cut-off Λ, a scale taken to be between M_{GUT} and M_{Pl}. At the scale Λ around 10–20 M_{GUT} the theory becomes strongly interacting. Note, Λ is large enough that the approximation of neglecting terms of order M_{GUT}/Λ is not unreasonable.

The missing partner mechanism to solve the doublet-triplet splitting problem occurs because the **50** contains a $(\bar{3}, 1)$, i.e. a colored anti-triplet and $SU(2)$ singlet with electric charge 1/3, but no colorless Higgs-like doublet (1,2). The $U(1)$ flavor symmetry prevents a μ term, $H\bar{H}$. Also no non-renormalizable terms of the form $H\bar{H}Y^m X^n$ $(m, n \geq 0)$ are possible, because X has a negative Q charge. Thus in this theory doublet-triplet splitting occurs naturally due to the absence of a Higgs doublet partner in the 50, $\overline{50}$. This is the so-called missing partner mechanism [73].

The Higgs color triplets mix with the analogous states in the 50 and the resulting mass matrix is of the see-saw form:

$$\hat{m}_T = \begin{bmatrix} 0 & c_2\langle Y\rangle \\ c_3\langle Y\rangle & c_4\langle X\rangle \end{bmatrix} \quad . \tag{9.4}$$

Defining $m_\phi = c_4\langle X\rangle$ the eigenvalues of the matrix $\hat{m}_T\hat{m}_T^\dagger$ are the squares of:

$$m_{T1,2} = \frac{1}{2}\left[\sqrt{m_\phi^2 + (c_2+c_3)^2\langle Y\rangle^2} \pm \sqrt{m_\phi^2 + (c_2-c_3)^2\langle Y\rangle^2}\right] \quad . \tag{9.5}$$

The effective mass that enters in the dimension 5 baryon and lepton number violating operators is given by

$$m_T^{eff} = \frac{m_{T1}m_{T2}}{m_\phi} = \frac{c_2c_3}{c_4}\frac{\langle Y\rangle^2}{\langle X\rangle} \quad . \tag{9.6}$$

We discussed nucleon decay in this model earlier (see Table 6.1). It turns out that in order to fit the observed value of $\alpha_3(M_Z)$, the effective color triplet mass in this model is given by $m_T^{eff} \approx 3 \times 10^{18}$ GeV (see Fig. 9.2, taken from [97]). This is then also preferred by the bound on the proton lifetime via $p \to K^+\,\bar{\nu}$. By keeping fixed all the remaining parameters they obtain a proton decay rate via dimension 5 operators in the range $8 \times 10^{31} - 3 \times 10^{34}$ years for the channel $p \to K^+\,\bar{\nu}$ and a rate between 2×10^{32} years and 8×10^{34} years for the channel $p \to \pi^+\,\bar{\nu}$. Since the heavy vector boson mass, M_X, is equal to 2.9×10^{16} GeV in the model, the dimension 6 operators provide a proton lifetime for the channel $p \to e^+\,\pi^0$ larger than 10^{36} years.

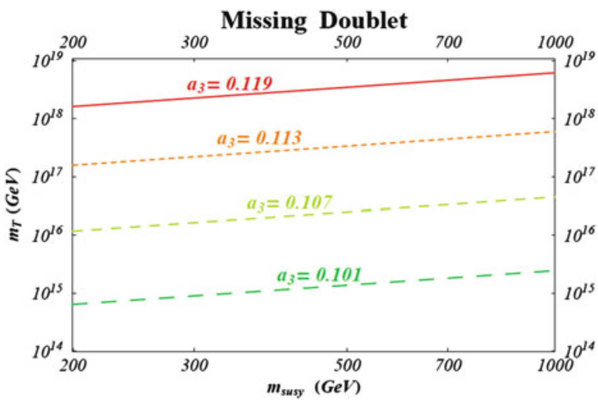

Fig. 9.2 The dependence of M_T^{eff} on the value of $\alpha_S(M_Z)$. *This figure is reproduced from Fig. 1, [97]*

A μ term is introduced into the model via the Giudice-Masiero mechanism [117] with a term in the Kähler potential given by

$$\mathscr{K} \supset \frac{S^\dagger\, X^\dagger\, H\, \bar{H}}{\Lambda^2} + h.c.. \tag{9.7}$$

With $\langle S \rangle \sim \theta^2\, m\, M_{Pl}$, $\langle X \rangle = \lambda\, \Lambda$, they obtain $\mu \sim \lambda\, m\, M_{Pl}/\Lambda$.[2]

9.1.1 Complete $SU(5)$ SUSY GUT: Yukawa Couplings

The $\mathscr{W}_3$ term contains the Yukawa interactions of the quark and lepton fields **10**, $\bar{\mathbf{5}}$ and **1**. They assumed an exact R-parity under which **10**, $\bar{\mathbf{5}}$ and **1** are odd whereas H, $\bar{H}$, H_{50} and $H_{\overline{50}}$ are even. The $\mathscr{W}_3$ term is symbolically given by

$$\begin{aligned} \mathscr{W}_3 = \mathbf{10}\, G_u(X,Y)\, \mathbf{10}\, H \;+\; \mathbf{10}\, G_d(X,Y)\, \bar{\mathbf{5}}\, \bar{H} \;+\; \bar{\mathbf{5}}\, G_\nu(X,Y)\, \mathbf{1}\, H \\ +M\, \mathbf{1}\, G_M(X,Y)\, \mathbf{1} \;+\; \mathbf{10}\, G_{\overline{50}}(X,Y)\, \mathbf{10}\, H_{\overline{\mathbf{50}}} \quad . \end{aligned} \tag{9.8}$$

The Yukawa matrices G_u, G_d, G_ν, G_M and $G_{\overline{50}}$ depend on X and Y and the associated mass matrices depend on their VEVs. The last term does not contribute to the mass matrices because of the vanishing VEV of $H_{\overline{50}}$, but it is important for proton decay. The pattern of fermion masses is determined by the $U(1)$ flavor symmetry that fixes the powers of $\lambda \equiv \langle X \rangle / \Lambda$ for each entry of the mass matrices.[3] In fact X is the only field with non vanishing Q that takes a VEV. The powers of λ in the mass terms are fixed by the Q charges of the matter fields and of the Higgs fields H and $\bar{H}$. They choose the Q charges of the matter fields in order to obtain realistic textures for the fermion masses.

$$Q(\mathbf{10}) = (4,3,1) \quad , \qquad Q(\bar{\mathbf{5}}) = (4,2,2) \quad , \qquad Q(\mathbf{1}) = (1,-1,0) \quad . \tag{9.9}$$

Then the Yukawa mass matrices are of the form:

$$G_r(\langle X \rangle, \langle Y \rangle)_{ij} \;=\; \lambda^{n_{ij}} G_r(\langle Y \rangle)_{ij}, \qquad r = u, d, \nu, M \quad . \tag{9.10}$$

We expand $G_r(\langle Y \rangle)_{ij}$ in powers of $\langle Y \rangle$ and consider the lowest order term at first. Taking $G_r(0)_{ij}$ of order 1 and n_{ij} as dictated by the above charge assignments we

[2] We can identify the field S as the Polonyi field which spontaneously breaks SUSY in a supergravity model. It then also generates all scalar masses. S has $U(1)$ charge, $Q_S = 0$.

[3] This is the $U(1)$ Froggatt-Nielsen mechanism [125].

obtain[4]:

$$m_u = \frac{1}{\sqrt{2}} \begin{bmatrix} \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{bmatrix} v_u \quad , \qquad m_d = m_e^T = \frac{1}{\sqrt{2}} \begin{bmatrix} \lambda^5 & \lambda^3 & \lambda^3 \\ \lambda^4 & \lambda^2 & \lambda^2 \\ \lambda^2 & 1 & 1 \end{bmatrix} v_d \lambda^4 \quad ,$$

$$m_\nu = \frac{1}{\sqrt{2}} \begin{bmatrix} \lambda^3 & \lambda & \lambda^2 \\ \lambda & 0 & 1 \\ \lambda & 0 & 1 \end{bmatrix} v_u \quad , \qquad m_{maj} = \begin{bmatrix} \lambda^2 & 1 & \lambda \\ 1 & 0 & 0 \\ \lambda & 0 & 1 \end{bmatrix} M \quad . \tag{9.11}$$

For a correct first approximation of the observed spectrum we need $\lambda \approx \lambda_C \approx 0.22$, λ_C being the Cabibbo angle. They have $\tan\beta = v_u/v_d \approx m_t/m_b \lambda^4$, which is small. The factor λ^4 is obtained as a consequence of the Higgs and matter field charges Q. Note a value of $\tan\beta$ near 1 is an advantage for suppressing proton decay.

The simple model as it now stands has order one parameters multiplying each term in the Yukawa matrices. Thus there are many arbitrary parameters, i.e. 18 for each 3 by 3 matrix. In addition it suffers from the $SU(5)$ mass relation, $\frac{m_s}{m_d} = \frac{m_\mu}{m_e}$. In order to fix this problem the authors add a term proportional to Y, the **75** dimensional field,

$$\begin{aligned} \mathscr{W}_3 = & \frac{1}{4} \mathbf{10}^{\alpha\beta}\, G_u\, \mathbf{10}^{\gamma\delta}\, H^\eta \epsilon_{\alpha\beta\gamma\delta\eta} + \frac{1}{4} \mathbf{10}^{\alpha\beta}\, G_{\overline{50}}\, \mathbf{10}^{\gamma\delta}\, H_{\overline{50}\alpha\beta\gamma\delta} \\ & + \sqrt{2}\, \mathbf{10}^{\alpha\beta}\, G_d\, \bar{\mathbf{5}}_\alpha\, \bar{H}_\beta + \frac{1}{\Lambda} \sqrt{2}\, \mathbf{10}^{\alpha\beta}\, F_d\, \bar{\mathbf{5}}_\gamma\, \bar{H}_\delta\, Y^{\gamma\delta}_{\alpha\beta} \\ & + \mathbf{1}\, G_\nu\, \bar{\mathbf{5}}_\alpha\, H^\alpha - \frac{1}{2} M\, \mathbf{1}\, G_M\, \mathbf{1} + \ldots \end{aligned} \tag{9.12}$$

where G_r $(r = u, d, \nu, M, \overline{50})$ is proportional to $G_r(\langle X \rangle, 0)$ of Eq. (9.8) and the coupling F_d is an $\langle X \rangle$-dependent matrix.

The term linear in Y in the previous equation is sufficient to differentiate the spectra in the charged lepton and down quark sectors. They get the following Dirac mass matrices:

$$m_{u,\nu} = y_{u,\nu} \frac{v_u}{\sqrt{2}} \quad , \quad m_{d,e} = y_{d,e} \frac{v_d}{\sqrt{2}} \quad , \tag{9.13}$$

$$\begin{aligned} & y_u = G_u \qquad\qquad y_\nu = G_\nu \quad , \\ & y_d = G_d + \frac{\langle Y \rangle}{\Lambda} F_d \qquad y_e^T = G_d - 3 \frac{\langle Y \rangle}{\Lambda} F_d \quad , \end{aligned} \tag{9.14}$$

[4]In our convention Dirac mass terms are given by $L^T m R^*$ and the light neutrinos effective mass matrix is $m_\nu m_{maj}^{-1} m_\nu^T$.

where v_u, v_d and $\langle Y \rangle$ parametrize the vevs of H, $\bar{H}$ and Y respectively. The fermion spectrum can be easily fit by appropriately choosing the numerical values of the matrices G_u, G_d, F_d, G_ν and G_M. Here is one set of values which, they claim, fits the data with $\lambda \equiv \langle X \rangle / \Lambda = 0.25$, $\tan\beta = 1.5$ and $M = 0.9 \cdot 10^{15}$ GeV.

$$G_u = \begin{bmatrix} (-0.51+0.61i)\lambda^6 & (0.42-0.70i)\lambda^5 & (0.27+0.86i)\lambda^3 \\ (0.42-0.70i)\lambda^5 & (-0.39+0.52i)\lambda^4 & (-0.30-1.14i)\lambda^2 \\ (0.27+0.86i)\lambda^3 & (-0.30-1.14i)\lambda^2 & 1.39 \end{bmatrix} , \tag{9.15}$$

$$G_d = \lambda^4 \begin{bmatrix} (2.39-1.11i)\lambda^5 & (0.33-0.59i)\lambda^3 & (0.13+0.45i)\lambda^3 \\ (0.87+0.55i)\lambda^4 & (2.76+0.89i)\lambda^2 & (0.69-0.51i)\lambda^2 \\ (-1.50+0.94i)\lambda^2 & (0.45+1.78i) & 1.94 \end{bmatrix} , \tag{9.16}$$

$$\frac{\langle Y \rangle}{\Lambda} F_d = \lambda^4 \begin{bmatrix} (0.38-0.18i)\lambda^5 & (-0.16-0.06i)\lambda^3 & (0.07+0.04i)\lambda^3 \\ (-0.08-0.05i)\lambda^4 & (-0.20-0.15i)\lambda^2 & (0.15-0.11i)\lambda^2 \\ (-0.09+0.12i)\lambda^2 & (0.07+0.14i) & -0.19 \end{bmatrix} , \tag{9.17}$$

$$G_\nu = \begin{bmatrix} (-0.78-0.19i)\lambda^3 & (0.52-0.34i)\lambda & (1.38+0.39i)\lambda^2 \\ (-1.23-0.34i)\lambda & 0 & (1.04+1.31i) \\ (0.45+1.18i)\lambda & 0 & 0.8+1.2i \end{bmatrix} , \tag{9.18}$$

$$G_M = \begin{bmatrix} (1.50+0.55i)\lambda^2 & (1.41+1.19i) & (0.35-1.53i)\lambda \\ (1.41+1.19i) & 0 & 0 \\ (0.35-1.53i)\lambda & 0 & 1.26+1.48i \end{bmatrix} . \tag{9.19}$$

9.2 Complete $SO(10)$ Model: GUT Breaking and Doublet-Triplet Splitting

This is the model introduced in [148]. A natural solution to the doublet-triplet [DT] splitting problem, avoiding severe fine-tuning is realized in SUSY $SO(10)$ by the so called Dimopoulos-Wilczek (or the missing VEV) mechanism [73, 74, 100]. It involves a coupling of two 10-plets of the form $H(10)\,A(45)\,H'(10)$ with the adjoint $A(45)$ having a GUT scale VEV in the $(B-L)$-preserving direction:

$$\langle A \rangle = \mathrm{i}\sigma_2\, \mathrm{Diag}\,(a,\ a,\ a,\ 0,\ 0)\ . \tag{9.20}$$

This structure contributes to the triplet and not to the doublet masses, and therefore can lead to natural DT splitting without fine-tuning.

In order to break SUSY $SO(10)$ to the supersymmetric standard model with a stabilized DT sector, and for the subsequent breaking of the electro-weak symmetry, the authors use a minimal low dimensional Higgs system. It consists of a single adjoint $A(45)$, two pairs of spinor-antispinor superfields $\{C(16) + \bar{C}(\overline{16})\}$ and

Table 9.3 $\mathscr{U}(1)_A$ and Z_2 charges Q_i and ω_i of the superfield ϕ_i

	$A(45)$	$H(10)$	$H'(10)$	$C(16)$	$\bar{C}(\overline{16})$	Z	S	$C'(16)$	$\bar{C}'(\overline{16})$	$16_{1,2}$	16_3
Q	0	1	-1	$\frac{9}{10}$	$-\frac{1}{2}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	$-\frac{13}{10}$	$-\frac{11}{10}$	$-\frac{1}{2}$
ω	1	0	1	0	0	1	0	0	0	$P_{1,2}$	0

The transformations under $\mathscr{U}(1)_A$ and Z_2 are respectively $\phi_i \to e^{iQ_i}\phi_i$ and $\phi_i \to e^{i\frac{2\pi}{2}\omega_i}\phi_i$

$\{C'(16) + \bar{C}'(\overline{16})\}$, two 10-plets $H(10)$ and $H'(10)$, as well as two $SO(10)$ singlets S and Z. The second spinorial pair $C' + \bar{C}'$ is introduced, following [149], to avoid pseudo-Goldstone degrees of freedom while maintaining the Dimopoulos-Wilczek VEV structure for A [cf: Eq. (9.20)]. Note, the spinor-antispinor superfields are introduced to spontaneously break $SO(10) \to SU(5)$, while the adjoint superfield, A, then breaks $SU(5) \to SU(3) \times SU(2) \times U(1)$. The S and Z superfields are needed to fix various VEVs in the required directions through their superpotential couplings.

The gauge symmetry is supplemented by a Z_2-assisted anomalous $\mathscr{U}(1)_A$ symmetry in order to stabilize the VEV pattern of Eq. (9.20) [150–153]. The charges of the Higgs fields and those of the three matter families 16_i under $\mathscr{U}(1)_A \times Z_2$ are listed in Table 9.3. The superpotential of the symmetry breaking sector, consistent with these symmetries, is $\mathscr{W} = \mathscr{W}_1 + \mathscr{W}_2 + \mathscr{W}_3$, where

$$\mathscr{W}_1 = M_A\,\mathrm{Tr}A^2 + \frac{\lambda_A}{M_*}\left(\mathrm{Tr}A^2\right)^2 + \frac{\lambda'_A}{M_*}\,\mathrm{Tr}A^4\,, \tag{9.21}$$

$$\mathscr{W}_2 = C\left(\frac{a_1}{M_*}\,Z\,A + \frac{b_1}{M_*}\,C\,\bar{C} + c_1 S\right)\bar{C}' + C'\left(\frac{a_2}{M_*}\,Z\,A + \frac{b_2}{M_*}\,C\,\bar{C} + c_2 S\right)\bar{C},$$

$$\mathscr{W}_3 = \lambda_1\,H\,A\,H' + \left(\lambda_{H'}\,S\,Z^4\right)\frac{(H')^2}{M_*^4} + \lambda_2\,H\,\bar{C}\,\bar{C} + \frac{\lambda_3}{M_*}\,A\,H'\,C\,C' + \frac{S\,Z^2\,C'\,\bar{C}'}{M^2}.$$

The $SO(10)$ contractions in the $C\bar{C}$ terms with coefficients $b_{1,2}$ are in the singlet channel. Matter parity is automatic, being part of $\mathscr{U}(1)_A$. The parities $P_{1,2}$ and charges of the first two families are relevant for the generation of quark and lepton masses.

Using the SUSY preserving condition $F_Z = F_S = 0$, together with the choice $\langle C\rangle = \langle\bar{C}\rangle = c$, $\langle A\rangle \neq 0$ (which is one allowed option among the discrete set of degenerate vacuum solutions), they get $\langle C\,\bar{C}'\rangle = \langle\bar{C}\,C'\rangle = 0$ and $\langle C'\rangle = \langle\bar{C}'\rangle = 0$. The VEV of A is then determined entirely by $\mathscr{W}_1$ of Eq. (9.21). Setting $F_A = 0$, they find a solution in the $B - L$ direction as in Eq. (9.20), with

$$a^2 = \frac{M_A\,M_*}{2(6\lambda_A + \lambda'_A)}\,. \tag{9.22}$$

With $\lambda_A, \lambda'_A \sim 1$, $M_* \sim 10^{18}$ GeV, and $M_A \sim 10^{15}$ GeV they obtain $a \sim M_{\mathrm{GUT}} \approx 2 \cdot 10^{16}$ GeV. Demanding F-flatness conditions $F_{C'} = F_{\bar{C}'} = 0$ they get $s = \langle S\rangle =$

$\frac{c^2}{M_*}\rho_1$ and $z = \langle Z\rangle = \frac{c^2}{3a}\rho_2$, where $\rho_1 = \frac{b_1a_2-b_2a_1}{a_1c_2-a_2c_1}$, $\rho_2 = \frac{b_1c_2-b_2c_1}{a_1c_2-a_2c_1}$. Note that for all dimensionless couplings in the Lagrangian being in the range $(1/4-2)$, the effective couplings $\rho_{1,2}$ can naturally take values as small as about $1/50$.

The sum of the VEVs gets further constrained as follows. The anomalous $\mathscr{U}(1)_A$ symmetry, presumed to have a string origin, generates the Fayet-Iliopoulos term ξ through quantum gravity, which is given by Dine et al. [154] $\xi = \frac{g_{st}^2 M_{Pl}^2}{192\pi^2}\mathrm{Tr}Q_A$, where g_{st} denotes the string coupling and $m_{Pl} \simeq 2.4\cdot 10^{18}$ GeV is the reduced Planck mass. In their model, the particle spectrum of Table 1 would lead to $\mathrm{Tr}(Q_A) = -84/5$. With the charges in Table 9.3, the vanishing of $D_A = \xi + \sum_i Q_i|\langle\phi_i\rangle|^2 = 0$ (required for preserving SUSY), yields $c^2+|z|^2+|s|^2 = -\frac{5}{2}\xi$. Thus, the VEVs of all the fields get determined. We see that quite naturally, the VEVs $c, z \sim (\text{few} - 10)\times M_G$, and $s \sim (10^{-2} - 10^{-1})\times M_G$ can arise, with the precise values depending on the order one couplings.

Substituting the VEVs of the heavy fields in Eq. (9.21), they obtain the mass matrices M_D and M_T for the $SU(2)_L$ doublets and $SU(3)_c$-color triplets (written in the $SU(5)$ notation):

$$M_{D,T} = \begin{array}{c} \\ \bar{5}_H \\ \bar{5}_{H'} \\ \bar{5}_C \\ \bar{5}_{C'} \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 5_H & 5_{H'} & 5_{\bar{C}} & 5_{\bar{C}'} \end{array} \\ \begin{pmatrix} 0 & \eta_{D,T}\lambda_1 a & \lambda_2 c & 0 \\ -\eta_{D,T}\lambda_1 a & M_{H'} & 0 & 0 \\ 0 & 0 & 0 & \kappa_{D,T}Y_1 \\ 0 & Y_{D,T} & \kappa_{D,T}Y_2 & M_{C'} \end{pmatrix} \end{array}, \tag{9.23}$$

with $(\eta_D, \eta_T) = (0,1)$, $(\kappa_D, \kappa_T) = (3,2)$. Here $M_{H'} = (\lambda_{H'}sz^4)/M_*^4$ (which is in the range $(10^{11} - 10^{12})$ GeV), $Y_{1,2} = 2a_{1,2}za/(M_*)$ and $Y_{D,T} \sim \lambda_3\langle A\rangle c/(M_*)$. The suppressed mass of H' is crucial for the adequate suppression of $d = 5$ proton decay. The entry $M_{C'}$ in Eq. (9.23) (allowed by the stability of Higgs doublet mass) arises from the operator $SZ^2C'\bar{C}'/M^2$ and yields $M_{C'} \sim (10^{-2}$ to $10^{-1})\times M_{\mathrm{GUT}}$ if $M \sim z$, which happens if the superfields that are integrated out have GUT scale masses.

The zeros in the first column of Eq. (9.23) are ensured, in the presence of *all higher dimensional operators*, for the doublet mass matrix by the $\mathscr{U}(1)_A \times Z_2$ symmetry. The main reason for this all-order stability of the Higgs doublet masses is that all the effective Higgs fields (i.e. any positive power of Z, S and $\bar{C}C$) which have super-large VEVs are positively charged under $\mathscr{U}(1)_A$, and can not couple to H^2, which is also positively charged. Thus, with $\eta_D = 0$ one pair of the Higgs doublets will be massless, while the remaining three pairs of doublets become superheavy. The role of the Z_2 symmetry is that it allows the coupling of H to H' only through A (or odd powers of A). Such couplings, however, do not generate a doublet mass due to the VEV structure in Eq. (9.20) of $\langle A\rangle$. The VEV pattern of $\langle A\rangle$ along the $B-L$ direction is also guaranteed to be stable because of the $\mathscr{U}(1)_A$ symmetry. Indeed, note that the symmetry $\mathscr{U}(1)_A$ does not allow any superpotential coupling involving A, C and $\bar{C}$ of the form $A^n(C\bar{C})^m$. It is only these couplings which, if allowed, would have upset the missing VEV pattern of Eq. (9.20). Their absence to all orders thus guarantees that the pattern of Eq. (9.20) is absolutely stable (barring of course SUSY breaking at the TeV scale which is safe). As far as the color-triplets are concerned, since $\eta_T \neq 0$ in Eq. (9.23) for the triplets, all four pairs become super-heavy, just as desired.

The two massless Higgs doublets which emerge from Eq. (9.23) represent the MSSM doublets h_u and h_d which acquire light masses after SUSY breaking. If one denotes the down type doublets in H, H', C and C' by H_d, $H_d{}'$, C_d and $C_d{}'$ respectively, and likewise the up-type doublets. It is easy to see from Eq. (9.23) that h_u is composed entirely of H—i.e. $h_u = H_u$, while h_d is a mixture of four components H_d, $H_d{}'$, C_d and $C_d{}'$ with $H \supset \cos\gamma \cdot h_d$, $H' \supset \frac{\lambda_2 c Y_D}{3 Y_2 M_{H'}} \cos\gamma \cdot h_d$, $C \supset \frac{\lambda_2 c M_{C'}}{9 Y_1 Y_2} \cos\gamma \cdot h_d$ and $C' \supset \frac{\lambda_2 c}{3 Y_2} \cos\gamma \cdot h_d$. The angle γ is determined in terms of the parameters of the superpotential. It is related to the MSSM parameter $\tan\beta$ as $\tan\beta = \frac{m_t}{m_b} \cos\gamma$. Note that, unlike in many $SO(10)$ models, the MSSM parameter $\tan\beta$ is not required to be large here.[5] It would turn out that conservative upper limits on proton lifetime correspond to smaller values of $\tan\beta$.

It can be shown that the effective low energy theory includes only the states of the MSSM. Therefore gauge coupling unification can be evaluated as usual. The effective color triplet mass defined by

$$\frac{1}{M_T^{eff}} = (M_T^{-1})_{11} = \frac{M_{H'}}{\lambda_1^2 a^2} \tag{9.24}$$

can have values in the range

$$M_T^{eff} \sim (5 \times 10^{16} - 6 \times 10^{19})\,\mathrm{GeV}, \tag{9.25}$$

depending on the choice of the arbitrary parameters. This affects proton decay via dimension five operators. In addition, the gauge bosons in $SO(10)/[SU(3) \times SU(2) \times U(1)]$ obtain mass of order $M_X = g_G\, a$. This affects proton decay via dimension 6 operators.

It turns out that requiring self-consistent gauge coupling unification (taking into account two loop RG running from the weak scale to the GUT scale and one loop threshold corrections at both the weak and GUT scales) gives a relation between M_T^{eff} and M_X. They find

$$\begin{aligned} M_T^{eff} &\simeq 10^{19}\,\mathrm{GeV} \left(\frac{10^{16}\,\mathrm{GeV}}{M_X}\right)^3 \left(\frac{3}{\tan\beta}\right) \left(\frac{0.01}{r}\right) \left(\frac{0.6}{\eta_\gamma}\right) \\ &\times \left\{ \frac{\exp[2\pi(\Delta^{(2)}_{2,\,w} - \Delta^{(2)}_{3,\,w} - \delta\alpha_3^{-1})]}{2.54 \cdot 10^{-2}} \right\} \,, \end{aligned} \tag{9.26}$$

[5]Note, this is a choice, since if $\lambda_2 = 0$, then $\cos\gamma = 1$ and $h_d = H_d$.

where

$$r = \frac{M_\Sigma}{M_X} = \frac{4\lambda'_A M_X}{g_G^2 M_*} \approx \left(\frac{1}{15} - \frac{1}{300}\right) \tag{9.27}$$

with $M_\Sigma = M_8 = 2M_3$ (i.e. the color octet mass, M_8, and $SU(2)$ triplet mass, M_3), $\eta_\gamma \simeq 0.6$ accounts for the running of $\cos\gamma$, and $\delta\alpha_3^{-1}$ denotes the deviation of α_3^{-1} from its central value of $1/0.1176$. Note that the curly bracket on the right side of Eq. (9.26) is fully determined for any given choice of the SUSY parameters and $\alpha_3(M_Z)$. This is only mildly dependent on variations of m_0 and $M_{1/2}$. In Fig. 9.3 (taken from [148]) we show the result of RG running.

Now, using expressions for proton decay rates one finds that the empirical lower limit on $\Gamma^{-1}(p \to \bar{\nu}K^+)$ requires that $M_T^{eff} \gtrsim 2.91 \cdot 10^{19}$ GeV (for reasonable scenarios for the Yukawa couplings), while that on $\Gamma^{-1}(p \to e^+\pi^0)$ requires [owing to Eq. (9.26)] $r \lesssim 1/150$. Using the particular choice of SUSY parameters stated above, and the ranges for M_T^{eff} and r as given in Eqs. (9.25) and (9.27), they illustrate the correlation between M_T^{eff} and M_X in Fig. 9.4 (taken from [148]) by confining to the ranges $M_T^{eff} \simeq (2.91-6)\times 10^{19}$ GeV and $r \simeq (1/200-1/300)$. Note, when M_X decreases, dimension 6 nucleon decay dominates, since M_T^{eff} increases and vice versa, when M_X increases, dimension 5 nucleon

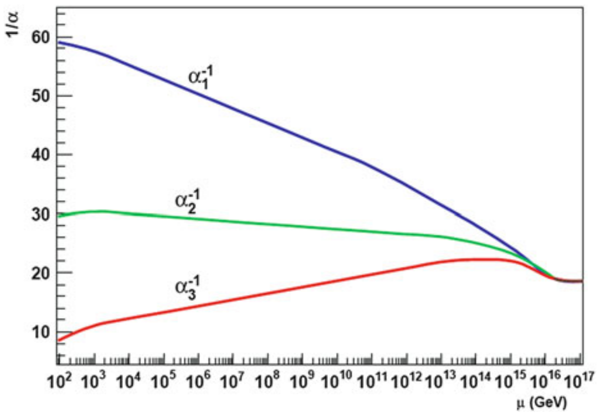

Fig. 9.3 Evolution of the three standard model gauge couplings in the present $SO(10)$ model including threshold corrections using $\alpha_3(M_Z) = 0.1176$ with CMSSM boundary conditions given by $\{\tan\beta, m_0, M_{1/2}, \mu\} = \{3, 1448.2\,\text{GeV}, 155.93\,\text{GeV}, 1\,\text{TeV}\}$ (corresponding to $m_{\tilde{q}} = 1.5$ TeV, $m_{\tilde{W}} = 130$ GeV), and with $r = 1/250$, $M_T^{eff} = 4\times 10^{19}$ GeV, $Y_{1,2} = 2M_X/45$ for generating this plot. *This figure is reproduced from Fig. 1 of Babu et al. [148]*

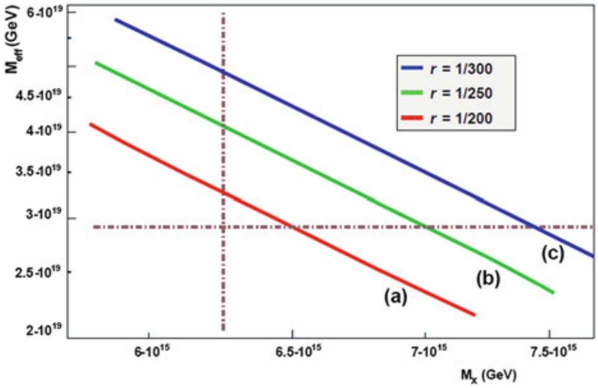

Fig. 9.4 Correlations between M_T^{eff} and M_X for $\{\tan\beta, m_0, M_{1/2}, \mu\} = \{3, 1448.2\,\text{GeV}, 155.93\,\text{GeV}, 1\,\text{TeV}\}$ (corresponding to $m_{\tilde{q}} = 1.5\,\text{TeV}$, $m_{\tilde{W}} = 130\,\text{GeV}$), and $\alpha_3(M_Z) = 0.1176$. (a) $r = 1/200$. (b) $r = 1/250$. (c) $r = 1/300$. The *vertical and horizontal dashed lines* correspond to the experimentally allowed lowest values of M_X and M_T^{eff} which arise from limits on $\Gamma^{-1}(p \to e^+\pi^0)$ and $\Gamma^{-1}(p \to \bar{\nu}K^+)$ respectively, for central values of the relevant parameters. *This figure is reproduced from Fig. 2 of Babu et al. [148]*

decay dominates, since M_T^{eff} decreases. Thus nucleon decay cannot be put off indefinitely!

9.2.1 Complete $SO(10)$ SUSY GUT: Yukawa Couplings

In order to calculate the proton decay rates via dimension 5 and 6 operators one needs to specify the Yukawa couplings which define the mass matrices and mixing angles. In order to present a predictive theory of fermion masses and mixing, the authors enlarge the symmetry groups to include a non-Abelian discrete flavor symmetry, Q_4 (the quaternionic group).[6] The matter fields $16_{1,2}$ transform as a doublet, $\overrightarrow{16} = (16_1, 16_2)$. 16_3 and the Higgs, H, transform as Q_4 singlets. Two Q_4 doublet flavon fields $\overrightarrow{X}$, $\overrightarrow{Y}$ both with VEVs along the $(1,0)$ direction are also utilized. The Q_4 symmetry also enables the authors to successfully address the SUSY FCNC problem [155]. With the $\mathcal{U}(1)_A$ charge assignments of $Q(\overrightarrow{16}) = -11/10$, $Q(\overrightarrow{X}) = 3/5$ and $Q(\overrightarrow{Y}) = 7/5$, the relevant operators, in accord with the symmetry $SO(10) \times \mathcal{U}(1)_A \times Z_2 \times Q_4$,[7] which generate effective Dirac Yukawa

[6] See problem 7.

[7] We note, the Z_2 symmetry is violated by some of the effective fermion mass operators independent of the Z_2 charges $P_{1,2}$. It can however be extended to a Z_4 symmetry which is then consistent with all terms.

couplings, are:

$$16_3\, 16_3\, H, \qquad \frac{\overrightarrow{X}}{M_*}\, \overrightarrow{16}\, 16_3\, H, \qquad \frac{SZ^2A}{M_*^4}\, \overrightarrow{16}\, \overrightarrow{16}\, H, \tag{9.28}$$

$$\frac{Z^3C}{M_*^4}\, \overrightarrow{16}\, \overrightarrow{16}\, C', \ \frac{AC\overrightarrow{Y}}{M_*\langle Z\rangle^2}\left(\overrightarrow{16}\cdot 16_3 + 16_3\cdot\overrightarrow{16}\right)C', \ \frac{AC}{M_*^2\langle Z\rangle^2}(\overrightarrow{X}\,\overrightarrow{16})(\overrightarrow{Y}\,\overrightarrow{16})C'.$$

The higher order operators, suppressed by powers of $1/M_*$, and in the last two cases by $1/\langle Z\rangle^2$ as well, may be generated by quantum gravity and in part by exchange of additional heavy vector-like states. Note, the resulting mass matrices for the quarks and charged leptons at the GUT scale have the form:

$$M_{u(\nu)} = \begin{array}{c} \\ u_1(\nu_1) \\ u_2(\nu_2) \\ u_3(\nu_3) \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \bar{u}_1(\bar{\nu}_1) & \bar{u}_2(\bar{\nu}_2) & \bar{u}_3(\bar{\nu}_3) \end{array} \\ \begin{pmatrix} 0 & \kappa_{u(\nu)}\epsilon' & 0 \\ -\kappa_{u(\nu)}\epsilon' & 0 & \sigma \\ 0 & \sigma & 1 \end{pmatrix} \end{array} m_U^0\,, \tag{9.29}$$

$$M_{d(e)} = \begin{array}{c} \\ d_1(\bar{e}_1) \\ d_2(\bar{e}_2) \\ d_3(\bar{e}_3) \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \bar{d}_1(e_1) & \bar{d}_2(e_2) & \bar{d}_3(e_3) \end{array} \\ \begin{pmatrix} 0 & \kappa_{d(e)}\epsilon' + \eta' & 0 \\ -\kappa_{d(e)}\epsilon' - \eta' & \kappa_{d(e)}\xi^d_{22} & \sigma + \kappa_{d(e)}\epsilon \\ 0 & \sigma + \kappa_{d(e)}\overline{\epsilon} & 1 \end{pmatrix} \end{array} m_D^0\,,$$

where $m_D^0 = m_U^0 \cos\gamma/\tan\beta$, $\kappa_{u,d} = 1$ and $\kappa_\nu = -3, \kappa_e = 3$. Equation (9.29) provide a constrained system with fewer parameters than observables. A consistent fit for all masses and mixing parameters as well as observed CP violation is obtained with the choice $\sigma = 0.0508,\ \epsilon = -0.0188 + 0.0333i,\ \overline{\epsilon} = 0.106 + 0.0754i,\ \epsilon' = 1.56\cdot 10^{-4},\ \eta' = -0.00474 + 0.00177i,\ \xi^d_{22} = 0.014e^{4.1i}$ at the GUT scale. Upon renormalization down to low energies (with $m_t(m_t) = 160\,\text{GeV}$ and $\tan\beta = 3$), these values reproduce the central values of the charged lepton masses. In addition, for the quark masses they obtain

$$\begin{aligned} &m_u(2\,\text{GeV}) = 3.55\,\text{MeV},\ m_c(m_c) = 1.15\,\text{GeV}, \\ &m_d(2\,\text{GeV}) = 6.45\,\text{MeV},\ m_s(2\,\text{GeV}) = 137.6\,\text{MeV},\ m_b(m_b) = 4.67\,\text{GeV}. \end{aligned} \tag{9.30}$$

Also for the CKM mixings they obtain at $\mu = M_Z$,

$$\begin{aligned} &|V_{us}| = 0.225\,,\ \ |V_{cb}| = 0.0414\,,\ \ |V_{ub}| = 0.0034\,,\ \ |V_{td}| = 0.00878, \\ &\overline{\eta} = 0.334\,,\ \ \overline{\rho} = 0.12\,. \end{aligned} \tag{9.31}$$

They find $\sin 2\beta = 0.663$. All these are in a reasonable agreement with experiments. The right-handed neutrino Majorana mass matrix is given by

$$M_{\bar{\nu}} = \begin{array}{c} \\ \bar{\nu}_1 \\ \bar{\nu}_2 \\ \bar{\nu}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \bar{\nu}_1 & \bar{\nu}_2 & \bar{\nu}_3 \end{array} \\ \begin{pmatrix} b & 0 & 0 \\ 0 & b & a \\ 0 & a & 1 \end{pmatrix} \end{array} M_0 \,, \tag{9.32}$$

They obtain the predictions in the neutrino sector

$$\sqrt{\Delta m^2_{sol}/\Delta m^2_{atm}} = m_2/m_3 = 0.13 \quad \text{and} \quad \theta_{13} = 3.6^\circ. \tag{9.33}$$

Given the fermion Yukawa matrices, much more analysis can be done checking the agreement with precision electroweak data and flavor violating processes at low energies. Nevertheless, the authors then analyze the predictions for the dominant proton decay rates which was input for Fig. 9.4.

9.3 Summary of Complete SUSY GUT Models

We have discussed two complete 4D SUSY GUT models with natural doublet-triplet splitting. Each has a GUT breaking sector and doublet-triplet splitting sector which contributes in a non-trivial way to gauge coupling unification and provides non-trivial constraints on nucleon decay. In both cases, the models are consistent with present bounds on the proton lifetime. In addition both models use the Froggatt-Nielsen mechanism with a $U(1)$ family symmetry for the $SU(5)$ construction and a discrete non-Abelian family symmetry for the $SO(10)$ construction to constrain fermion Yukawa coupling matrices and thus fit, at low energies, fermion masses and mixing angles. It is clear that in the case of a $U(1)$ family symmetry there are still more free parameters than observables. So there are no predictions at low energies. Nevertheless, once one fits the fermion masses and mixings, there are predictions for nucleon decay rates and branching ratios. In general, however, a $U(1)$ family symmetry cannot by itself ameliorate the SUSY flavor and CP problems. In the case of the non-Abelian family symmetry, there are more constraints on the fermion Yukawa couplings (in fact, in the charged fermion case there are only 10 arbitrary Yukawa parameters plus $\tan\beta$ to fit the 13 low energy observables) and thus the theory is more predictive.

Finally, it is easy to see that there are two, almost independent, sets of predictions. The GUT breaking and doublet-triplet splitting sectors control threshold corrections at the GUT scale which directly affect gauge coupling unification and the magnitude of the proton lifetime. On the other hand, the GUT symmetry combined with a flavor symmetry controls the Yukawa matrices and thus direct low energy tests of the theory—at colliders and astrophysical consequences. In the next section, we will focus on the latter consequences of a SUSY GUT. We discuss an $SO(10)$ SUSY GUT which has been thoroughly tested with low energy data. This may not be the theory of the universe, but nevertheless we show how any such theory can be fully tested by data.

Chapter 10
$SO(10)$ SUSY GUT and Low Energy Data

10.1 Yukawa Coupling Unification

Gauge coupling unification in supersymmetric grand unified theories (SUSY GUTs) [29, 51, 72, 75–77] provides an experimental hint for low energy SUSY. However, it does not significantly constrain the spectrum of supersymmetric particles. On the other hand, it has been observed that Yukawa coupling unification for the third generation of quarks and leptons in models, such as SO(10) or $SU(4)_c \times SU(2)_L \times SU(2)_R$, can place significant constraints on the SUSY spectrum in order to fit the top, bottom and tau masses [156–160]. These constraints depend on the particular boundary conditions for sparticle masses chosen at the GUT scale.[1] In light of the present success of the LHC with the observation of the Higgs boson with mass of order 125 GeV and significant lower bounds on gluino and squark masses, it is a perfect time to review the viability of the constraints on the sparticle spectrum resulting from gauge and third generation Yukawa coupling unification.[2] This is what we do in this chapter. In the first part of the chapter we demonstrate the constraints on the sparticle spectrum coming just by considering the third family. We perform a global χ^2 analysis assuming SO(10) boundary conditions for sparticle masses and non-universal Higgs masses, which we have called "just so Higgs splitting." We fit the observables, $M_W, M_Z, G_F, \alpha_{em}^{-1}$, $\alpha_s(M_Z), M_t, m_b(m_b), M_\tau, BR(B \rightarrow X_s\gamma), BR(B_s \rightarrow \mu^+\mu^-)$ and M_h in terms of 11 (or 12) arbitrary parameters. These fits then place significant constraints on the upper bound for the gluino mass. As we will discuss later the LHC determined lower bound on the gluino mass in these models is found to be of order 1.2 TeV

[1] For a discussion of Yukawa coupling unification with different boundary conditions than discussed here, see for example, [158, 161–164].

[2] For other analyses in this direction, see [165, 166].

S. Raby, *Supersymmetric Grand Unified Theories*, Lecture Notes in Physics 939,
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[167]. Note, the analysis in this chapter is heavily dependent on more recent work with my collaborators, see [167–170].

10.2 Third Family Model

Fermion masses and quark mixing angles are manifestly hierarchical. The simplest way to describe this hierarchy is with Yukawa matrices which are also hierarchical. Moreover the most natural way to obtain the hierarchy is in terms of effective higher dimension operators of the form

$$\mathscr{W} \supset \lambda\ 16_3\ 10\ 16_3 + 16_3\ 10\ \frac{45}{M}\ 16_2 + \cdots . \tag{10.1}$$

This version of SO(10) models has the nice features that it only requires small representations of SO(10), has many predictions and can, in principle, find a UV completion in string theory. The only renormalizable term in $\mathscr{W}$ is $\lambda\ 16_3\ 10\ 16_3$ which gives Yukawa coupling unification

$$\lambda = \lambda_t = \lambda_b = \lambda_\tau = \lambda_{\nu_\tau} \tag{10.2}$$

at M_{GUT}. Note, one *cannot* predict the top mass due to large SUSY threshold corrections to the bottom and tau masses, as shown in [171–174]. These corrections are of the form (see Fig. 10.1 for the dominant corrections)

$$\delta m_b/m_b \propto \frac{\alpha_3\ \mu\ M_{\tilde{g}}\ \tan\beta}{m_{\tilde{b}}^2} + \frac{\lambda_t^2\ \mu\ A_t\ \tan\beta}{m_{\tilde{t}}^2} + \text{log corrections.} \tag{10.3}$$

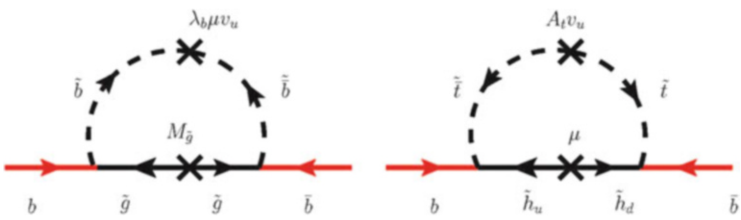

Fig. 10.1 The Feynman graphs corresponding to the dominant large $\tan\beta$ corrections to the bottom quark mass

So instead we use Yukawa unification to predict the soft SUSY breaking masses. In order to fit the data, we need[3]

$$\delta m_b/m_b \sim -2\%. \tag{10.4}$$

We take μ, $M_{\tilde{g}} > 0$, thus we need $A_t < 0$. For a short list of references on this subject, see [156–160, 165, 175–177].

The $SO(10)$ GUT models with universal and non-universal gaugino masses are both defined by three gauge parameters, $\alpha_G, M_G, \epsilon_3$; one large Yukawa coupling, λ; μ, and $\tan\beta$ are obtained at the weak scale by consistent electroweak symmetry breaking. There are 5 soft SUSY breaking parameters defined at the GUT scale: m_{16} (universal scalar mass for squarks and sleptons), $M_{1/2}$ (universal gaugino mass), A_0 (universal trilinear scalar coupling), and m_{H_u}, m_{H_d} (up and down Higgs masses).[4] The models with non-universal gaugino masses determined by "mirage mediation" [178–181] have one additional parameter in the SUSY sector, α. We have

$$M_i = \left(1 - \frac{g_G^2 b_i \alpha}{16\pi^2} \log\left(\frac{M_{Pl}}{m_{16}}\right)\right) M_{1/2} \;, \tag{10.5}$$

where $b_i = (-33/5, -1, 3)$ for $i = 1, 2, 3$, $M_{1/2}$ is the overall mass scale, and α is the ratio of the anomaly mediation to gravity mediation contributions. The size of α plays a crucial role in determining the ratio of the gaugino masses and in addition the spectrum that is consistent with Yukawa unification. Note, this expression is equivalent to the gaugino masses defined in [182]. α in the above expression is related to the ρ in [181] as: $\frac{1}{\rho} = \frac{\alpha}{16\pi^2} \ln \frac{M_{PL}}{m_{16}}$.

Given the GUT scale boundary conditions, we find that fitting the top, bottom and tau masses forces us into the region of SUSY breaking parameter space with[5]

$$A_0 \approx -2m_{16}, \quad m_{10} \approx \sqrt{2}\, m_{16}, \quad m_{16} > \text{ few TeV}, \quad \mu, M_{1/2} \ll m_{16}; \tag{10.6}$$

and, finally,

$$\tan\beta \approx 50. \tag{10.7}$$

[3] Note, in 1993 in [70] we used Yukawa unification and the observed bottom and tau masses to predict the top quark mass. We found $M_t = 180 \pm 15$ GeV with $\tan\beta \approx 50$, where the error was due to the large uncertainty in the value of $\alpha_s(M_Z)$. At the time we didn't know about the large threshold corrections to the bottom quark mass proportional to $\tan\beta$.

[4] We have non-universal Higgs masses which are necessary for robust radiative electroweak symmetry breaking. Note, some authors have also considered the contribution of a $U(1)_X$ D-term. In problem 8 we show that such a D-term decouples from the effective low energy theory assuming $U(1)_X$ is broken at the GUT scale.

[5] This is true as long as $\alpha \lesssim 2.5$ [169].

This result has been confirmed by several independent analyses [159, 160, 177].[6] Although the condition, Eq. (10.6), is not obvious, it is however easy to see that Eq. (10.7) is simply a consequence of third generation Yukawa unification, since $m_t(m_t)/m_b(m_t) \sim \tan\beta$. In addition, radiative electroweak symmetry breaking requires $\Delta^2_{m_H} = \frac{m^2_{H_d} - m^2_{H_u}}{2m^2_{10}} \approx 13\%$, with roughly half of this coming naturally from the renormalization group running of neutrino Yukawa couplings from M_G to $M_{N_\tau} \sim 10^{13}$ GeV [158].

It is very interesting that the above region in soft SUSY breaking parameter space results in an inverted scalar mass hierarchy [ISMH] at the weak scale with the third family scalars significantly lighter than the first two families [184]. These results depend solely on SO(10) Yukawa unification for the third family.[7] An ISMH has several virtues.

1. It preserves "naturalness" (for values of m_{16} which are not too large), since only the third generation squarks and sleptons couple strongly to the Higgs.
2. It ameliorates the SUSY CP and flavor problems, since these constraints on CP violating angles or flavor violating squark and slepton masses are strongest for the first two generations, yet they are suppressed as $1/m^2_{16}$. For $m_{16} >$ a few TeV, these constraints are weakened [104, 128, 129, 185].
3. Super-Kamiokande bounds on $\tau(p \to K^+\bar{\nu}) > 3.9 \times 10^{33}$ years [101] constrain the contribution of dimension 5 baryon and lepton number violating operators. These are however minimized with $\mu,\ M_{1/2} \ll m_{16}$ [98, 99].

10.2.1 Procedure for Third Family Analysis

Renormalization Group Equations

The model parameters, summarized in Table 10.1, are defined at the grand unification scale M_G with the exception of $\tan\beta$ and μ that are defined at the electroweak scale. At the GUT scale, $\alpha_G \equiv \alpha_1(M_G) = \alpha_2(M_G)$ and $\alpha_3(M_G) = \alpha_G(1 + \epsilon_3)$, where ϵ_3 is the GUT scale threshold correction[8] necessary to fit the strong coupling to experimental data at the electroweak scale, M_Z. In the third family analysis with 3 gauge parameters, 1 Yukawa coupling, 5 SUSY boundary conditions, $\tan\beta$ and μ, i.e. 11 arbitrary parameters (or 12 with α), (the right-handed neutrinos are integrated out at the GUT scale) we fit the 11 observables, $M_W, M_Z, G_F, \alpha^{-1}_{em}$,

[6] Note, different regions of parameter space consistent with Yukawa unification have also been discussed in [159, 160, 183].

[7] The large Yukawa coupling for the third family is the driving force for the inverted scalar mass hierarchy. However, the particular boundary conditions of Eq. (10.6) were shown to maximize the effect.

[8] Without presenting a complete GUT we leave ϵ_3 as a free parameter. In this way, our analysis will also apply to orbifold GUTs or string compactifications with a scale of order M_G.

Table 10.1 The $SO(10)$ GUT models with universal and non-universal gaugino masses are both defined by three gauge parameters, $\alpha_G, M_G, \epsilon_3$; one large Yukawa coupling, λ; μ, and $\tan\beta$ are obtained at the weak scale by consistent electroweak symmetry breaking

Sector	Universal gaugino masses	#	Non-universal gaugino masses	#
Gauge	$\alpha_G, M_G, \epsilon_3$	3	$\alpha_G, M_G, \epsilon_3$	3
SUSY (GUT scale)	$m_{16}, M_{1/2}, A_0, m_{H_u}, m_{H_d}$	5	$m_{16}, M_{1/2}, \alpha, A_0, m_{H_u}, m_{H_d}$	6
Textures	λ	1	λ	1
SUSY (EW scale)	$\tan\beta, \mu$	2	$\tan\beta, \mu$	2
Total #		11		12

There are five SUSY parameters defined at the GUT scale: m_{16} (universal scalar mass for squarks and sleptons), $M_{1/2}$ (universal gaugino mass), A_0 (universal trilinear scalar coupling), and m_{H_u}, m_{H_d} (up and down Higgs masses). The models with non-universal gaugino masses have one additional parameter in the SUSY sector, α, which is the ratio of anomaly mediation to gravity mediation contribution to gaugino masses

$\alpha_s(M_Z), M_t, m_b(m_b), M_\tau, BR(B \to X_s\gamma), BR(B_s \to \mu^+\mu^-)$ and M_h. Note, the two flavor violating observables are included since they severely constrain the theory in the large $\tan\beta$ limit. In order to evaluate the flavor violating observables, in this third family analysis, we use the relevant observed CKM matrix elements.

We use the 2-loop MSSM RGEs for both dimensionful and dimensionless parameters. Ideally, one should evolve all parameters to the scale of the heavy scalars (m_{16} in this case) and integrate them out and proceed to evolve to the weak scale using an effective theory without the first two generation scalars. We choose an alternative approach and use the 2-loop MSSM RGE[9] evolution down to the weak scale and correct for the additional running by including 1-loop threshold corrections to the relevant observables.[10] This approximation eliminates the need to define multiple effective theories. In our analysis, we have been careful to take into account the corresponding threshold corrections for all observables.

Electroweak Observables

At the weak scale, we calculate the SUSY spectrum and the SUSY threshold corrections to the fermion masses. Especially in the large $\tan\beta$ regime, these SUSY threshold corrections are very important for the down type quarks and charged leptons and can be at the 10 percent level in Yukawa-unified SUSY models [173, 174]. We then use the threshold corrected fermion masses to determine the tree level masses for the squarks and sleptons. In addition, we also determine the

[9]In scenarios with heavy scalars, it has been shown that the 2-loop contributions to the third generation scalars can lead to dramatic consequences, like driving the stop mass squared negative [186] and thus it is important to include the 2-loop RGEs in scenarios such as discussed here.

[10]For the calculation of Higgs mass, we define an effective theory at the scale M_{SUSY} and interface our calculation with the code by authors in [187].

one-loop pole mass for the gluino and the CP-odd Higgs mass. The precision electroweak observables M_Z, M_W, G_μ, $\alpha_{em}^{-1}(M_Z)$, $\alpha_s(M_Z)$ are calculated including 1-loop threshold corrections, using the procedure described in [188, 189]. Following the prescription in [188], the condition for consistent radiative electroweak symmetry breaking is also imposed at the weak scale, and for this, we use the physical Z pole mass. The parameter μ is fixed by this procedure via a separate χ^2 minimization, and in the process, we fit the Z mass precisely to the physical Z pole mass. In the calculation of M_Z and M_W, we only include the 1-loop corrections from the third family scalars, since the first two generation scalars are integrated out at m_{16}. We assign a theoretical uncertainty of 0.5% to our calculation of the electroweak observables (except for M_Z) due to the approximate treatment of thresholds described above. We also assign a 1% theoretical uncertainty to our calculation of G_μ, since we neglect the SUSY vertex and box diagrams. Finally, to compare to experiment, α_{em} is evolved to zero momentum transfer.

Charged Fermion Masses and Mixing Angles

Below M_Z, we integrate out all SUSY partners and electroweak gauge bosons to obtain an effective $SU(3) \times U(1)_{EM}$ low energy theory. We use 1-loop QED and 3-loop QCD RGEs to renormalize to the appropriate scales and calculate the low energy observables. We fit the top quark pole mass, and the bottom quark $\overline{\text{MS}}$ mass is calculated at its respective mass. The τ pole mass is calculated with 1-loop electromagnetic threshold corrections.

To execute the steps elaborated so far, we use a code `maton` , originally developed by Radovan Dermíšek to study Yukawa unification in the $SO(10)$ model with $D_3 \times [U(1) \times \mathbb{Z}_2 \times \mathbb{Z}_3]$ family symmetry [190]. `maton` has been restructured and extended appropriately (by Archana Anandakrishnan, B. Charles Bryant, Zijie Poh, and Akin Wingerter) to adapt to the current analysis.

Higgs Mass

The observation of the Higgs boson at the LHC [191, 192] severely constrains the parameter space of the model. Flavor constraints have already pushed the mass of the first two generation scalars of Yukawa-unified SUSY models to values $\gtrsim 10$ TeV [193]. In contrast, the third family scalars have mass about a few TeV, purely by the effects of RGE running. In addition to the TeV range scalars, the large A-terms, Eq. (10.6), make it easy to obtain a Higgs mass of about 125 GeV. We integrate out all the scalars (including the third generation squarks and sleptons) below the scale M_{SUSY}, and calculate the Higgs boson mass using the dedicated code by the authors of [187], that is best suited to our case where the sfermions are very heavy. Given the boundary conditions,

$$\mu(M_Z),\ M_1(M_{\text{SUSY}}),\ M_2(M_{\text{SUSY}}),\ M_3(M_{\text{SUSY}}),\ M_{\text{SUSY}},\ \tan\beta,\ A_t(M_{\text{SUSY}})$$

at the scale $M_{\mathrm{SUSY}} = \sqrt{M_{\tilde{t}_1} \times M_{\tilde{t}_2}}$, (where $M_1,\ M_2,\ M_3$ are the gaugino masses at the scale M_{SUSY}), the routine [187] determines the Higgs mass by calculating the corrections to the Higgs quartic coupling:

$$M_h = \sqrt{\frac{\lambda(M_{\mathrm{SUSY}})}{\sqrt{2}G_\mu}}\left(1 + \delta^{SM}(M_{\mathrm{SUSY}}) + \delta^{\chi}(M_{\mathrm{SUSY}})\right) \tag{10.8}$$

δ^{χ} are the contributions from chargino and neutralino diagrams. The quartic coupling $\lambda(m_{16})$ is given by:

$$\begin{aligned}\lambda(M_{\mathrm{SUSY}}) &= \frac{1}{4}\left(g^2(M_{\mathrm{SUSY}}) + g'^2(M_{\mathrm{SUSY}})\right)\cos^2 2\beta \\ &+ \frac{3\lambda_t^4}{8\pi^2}\left[\left(1 - \frac{g^2 + g'^2}{8\lambda_t^2}\right)\frac{X_t^2}{M_{\mathrm{SUSY}}^2} - \frac{X_t^4}{12M_{\mathrm{SUSY}}^4}\right]\end{aligned} \tag{10.9}$$

where $X_t = A_t - \mu/\tan\beta$. The parameters of the Lagrangian in the effective theory with the scalars integrated out are then evolved to a common renormalization scale M_Z by means of the one-loop RGEs of the effective low energy theory. Since some of the boundary conditions on the parameters are given at the SUSY scale, M_{SUSY}, and the others are given at the weak scale, M_Z, an iterative procedure is necessary. The resulting couplings evaluated at the weak scale account for the all-order resummation of the leading logarithmic corrections involving powers of $\log(M_{\mathrm{SUSY}}/M_Z)$.

We have to point out an important difference in our approach. The conventional method is a bottom-up approach which uses the SM inputs of M_Z, G_μ, $\alpha_{em}^{-1}(M_Z)$, $\alpha_s(M_Z)$, M_t, $m_b(m_b)$, M_τ to determine the gauge and the Yukawa couplings at the scales M_{SUSY} and further constrain the GUT scale parameters. Gauge coupling unification is obtained iteratively by varying parameters at the weak scale. We instead like to predict these low energy observables and constrain the GUT scale parameter space based on a global χ^2 fit to the data. In our calculation of the Higgs mass, we take the gauge and Yukawa couplings as input at the scale M_{SUSY}, obtained from RGE evolution using `maton` and calculate the Higgs mass using these inputs. The approach we adopt here is purely top-down. We have adapted the routine [187] to suit this line of analysis. Nevertheless, we have compared the spectrum we obtain from `maton` with that from `softsusy`[11] [194] and find good agreement.

[11]Without making significant changes to `softsusy` or other publicly available codes, we find that we can only make rough comparisons of the spectra. This is because to the best of our knowledge, most of the currently available codes do not handle complex parameters. In addition, many do not include right-handed neutrinos, and do not offer an easy way to implement the particular GUT scale Yukawa texture of the model.

Global Fit

In the last step of our calculation, we construct a χ^2 function in terms of the 11 calculated observables.

$$\chi^2 = \sum_i \frac{|y_i - y_i^{data}|^2}{\sigma_i^2} \tag{10.10}$$

y_i and y_i^{data} are the theoretical prediction and experimental measurement, respectively, for each observable. σ_i is the error on each observable, the theoretical and experimental errors added in quadrature. We consider the χ^2 for the model as a qualitative measure of the goodness of fit. Note that by good fits we refer to $\chi^2/d.o.f.$ less than 1.0, 2.6, and 3.8 at 68%, 90%, and 95% confidence levels, respectively. The number of degrees of freedom ($d.o.f.$) is defined to be the difference between the number of observables used in the fit and the number of parameters that are allowed to vary in the analysis. In the analysis we keep three parameters, m_{16}, $M_{1/2}$, α, fixed and vary the others. Thus in each third family analysis we have 2 $d.o.f.$ We minimize the χ^2 function using the Minuit package maintained by CERN [195]. Note that Minuit is not guaranteed to find the *global* minimum, but will in most cases converge on a local one. For that reason, we iterate $\mathcal{O}(100)$ times the minimization procedure for each set of input parameters, and in each step we take a different initial guess for the minimum (required by Minuit) so that we have a fair chance of finding the true minimum. This, of course, requires large computing resources, and to that end we have used the Ohio Supercomputer Center in Columbus. Note, when calculating flavor violating observables, we use `susy_flavor` with the experimental input values for the light fermion masses and mixing angles.

10.2.2 Results of Third Family Analysis

Consider first the SUSY spectrum in our analysis (see Table 10.2, taken from [167]). The first and second family squarks and sleptons have mass of order m_{16}, while stops, sbottoms and staus are all significantly lighter. This is the inverted scalar mass hierarchy which is a direct result of RG running. Nevertheless, gluinos are always lighter than the third family squarks and sleptons, and the lightest charginos and neutralinos are even lighter. Recent results from CMS and ATLAS give lower bounds on the gluino mass. These bounds are given in terms of the CMSSM or simplified models. The simplified models which are most relevant for our analysis are those in which (a) the third family of squarks and sleptons are lighter than the first two, and (b) the gluino is lighter than the stops and sbottoms. In this case, the lower bound on the gluino mass is now of order 1.3–1.4 TeV, *assuming* the branching ratio $BR(\tilde{g} \to t\bar{t}\,\tilde{\chi}_1^0) = 100\%$ or $BR(\tilde{g} \to b\bar{b}\,\tilde{\chi}_1^0) = 100\%$. However, in

Table 10.2 Benchmark points of the universal and non-universal gaugino masses

Benchmark model	Ud	Ue	Uf	DMa	DMb
ϵ_3	~ -0.03	~ -0.03	~ -0.03	0.00	0.00
α	0	0	0	1.5	2.3
$M_{1/2}$	300	400	600	450	600
μ	879	924	974	660	1199
m_{16}	20,000	20,000	20,000	20,000	29,781
χ^2/d.o.f.	0.58	1.12	1.98	0.92	0.86
M_A	2300	2361	2751	1915	3093
$M_{\tilde{g}}$	1187	1309	1430	1130	1135
$M_{\tilde{t}_1}$	3728	3760	3776	3612	5832
$M_{\tilde{b}_1}$	4608	4640	4628	4770	7543
$M_{\tilde{\tau}_1}$	7861	7896	7890	6867	10,565
$M_{\tilde{\chi}_1^0}$	195	217	239	474	799
$M_{\tilde{\chi}_2^0}$	382	422	461	557	836
$M_{\tilde{\chi}_3^0}$	882	927	977	663	1201
$M_{\tilde{\chi}_4^0}$	887	932	982	694	1211
$M_{\tilde{\chi}_1^+}$	382	422	461	555	836
$M_{\tilde{\chi}_2^+}$	888	933	982	691	1210
Ωh^2				0.121	0.099

The benchmark points have been chosen with varying values of gluino mass. The model provides good fits to all low-energy observables at 95% C.L.s for $M_{\tilde{g}} < 2000$ GeV (see Fig. 10.3). In the benchmark points for models consistent with Yukawa unification with small non-universal contributions the gaugino spectrum becomes increasingly compressed with increasing α. The well-tempered dark matter points were obtained by fixing μ, $M_{1/2}$ and α to get the right admixture for the LSP. The above fits were obtained by fixing m_{16} and $M_{1/2}$ (to obtain unique gluino masses) and thus yielding 2 d.o.f. for this system

Table 10.3 Gluino decay branching ratios into different final states for the five benchmark models given in Table 10.2

Benchmark model	Ud (%)	Ue (%)	Uf (%)	DMa (%)	DMb (%)
$BR(\tilde{g} \to t\bar{t}\tilde{\chi}_1^0)$	7	8	8	2	0
$BR(\tilde{g} \to t\bar{t}\tilde{\chi}_2^0)$	14	15	15	4	0
$BR(\tilde{g} \to b\bar{b}\tilde{\chi}_1^0)$	3	3	3	4	14
$BR(\tilde{g} \to b\bar{b}\tilde{\chi}_2^0)$	13	12	11	9	38
$BR(\tilde{g} \to tb\tilde{\chi}_1^{\pm})$	60	58	56	32	42
$BR(\tilde{g} \to tb\tilde{\chi}_2^{\pm})$	0	2	2	20	0

For each model, we give the dominant branching fractions. These ratios were calculated using `SDECAY` [201]

our case when $\alpha = 0$ the usual simplified models do not apply, since gluinos decay with branching ratios $\tilde{g} \to t\bar{t}\ \tilde{\chi}^0_{(1,2)}$, $b\bar{b}\ \tilde{\chi}^0_{(1,2)}$, $t\bar{b}\ \tilde{\chi}^-_{(1,2)}$, $b\bar{t}\ \tilde{\chi}^+_{(1,2)}$, $g\ \tilde{\chi}^0_{(1,2,3,4)}$ which are all significant (see Table 10.3).

When $\alpha < 4$, there are small non-universal contributions to the gaugino masses. The additional degree of freedom, α, allows tuning of the ratios of M_1 and M_2. As α is gradually increased from 0, there are two effects to the spectrum. The first one is that the wino component of the LSP begins to increase. Therefore, it is possible to obtain the correct relic dark matter density by tuning $M_{1/2}$, α, and μ. Secondly, since the beta-function coefficient is negative for SU(3), the gluino mass decreases with increasing α until it becomes the lightest supersymmetric particle for $\alpha \gtrsim 3$. The region in the parameter space that fits the measured value of dark matter relic abundance is of special interest and was studied in [196] (see also [181, 197, 198]). We show two sample benchmark models which reproduce the measured relic abundance [199] in Table 10.2. Point DMa is a well-tempered bino-wino-higgsino mixture whereas point DMb has a bino-wino mixture. The spin independent scattering cross-section for the benchmark point DMa is 1.6×10^{-8} pb. Note that this benchmark point is now ruled out by LUX results [200], which exclude spin independent scattering cross-sections down to 6×10^{-9} pb. An LSP of higher mass can be obtained by increasing the values of $M_{1/2}$ and μ while maintaining the relic density with bino-wino-higgsino well-tempering and be consistent with the LUX result. On the other hand, the spin independent scattering cross section for point DMb is below the bounds from LUX at 3.5×10^{-9} pb. The dominant decay modes of the gluino for points DMa and DMb are also very similar to the universal case with large decay fractions into the 3-body modes: $tb\tilde{\chi}^{\pm}$, $t\bar{t}\tilde{\chi}^0$, $b\bar{b}\tilde{\chi}^0$ (see Table 10.3).

In Fig. 10.3 we present χ^2 as a function of the gluino mass. More specifically, of the 11 parameters defining the third-family SO(10) Yukawa unified model (with $\alpha = 0$), we fix $m_{16} = 20$ TeV, and then minimize χ^2 for 15 values of $M_{1/2}$ between 0 and 600 GeV by varying all the other parameters. The x axis in Fig. 10.3 is $M_{\tilde{g}}$. The data points are indicated in Fig. 10.3 by crosses, and we have connected them using a cubic spline interpolation to guide the eye. We calculate the 68%, 90% and 95% confidence level intervals using the χ^2 distribution for 2 degrees of freedom; the corresponding values are indicated in Fig. 10.3 by the horizontal red lines. The simple explanation for this result is that as the gluino mass increases the magnitude of A_t at $M_{\rm SUSY}$ also increases, due to the infra-red fixed point. This has the effect of decreasing the light Higgs mass because now $X_t > \sqrt{6}M_{\rm SUSY}$ which goes beyond maximal mixing (see Fig. 10.2). As a consequence, there appears to be an upper bound on the gluino mass of order 2 TeV, which makes gluinos inevitably observable at the LHC 14 TeV.[12]

In [167] we evaluated the lower bound on the gluino mass by re-analyzing ATLAS and CMS data and making full use of the code CHECKMATE [204]. The procedure of [168] was performed for each benchmark model considered

[12] If we increase m_{16} to 30 TeV, the upper bound on the gluino mass is now 2.8 TeV. Therefore for $m_{16} = 25$ TeV we expect $M_{\tilde{g}} \lesssim 2.4$ TeV. Still observable at LHC II, but with less than a 5 σ discovery. CMS and ATLAS can obtain a 5 σ discovery of gluinos with mass less than $\sim$2 TeV with 300 fb^{-1} of data [203].

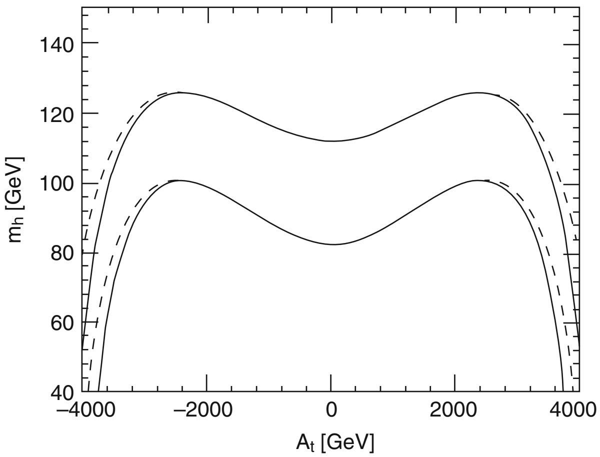

Fig. 10.2 Higgs mass plotted as a function of A_t, taken from Fig. 2, [202]. The value of the Higgs mass increases as $\tan\beta$ increases. *Reprinted from Nuclear Physics B 461, M. Carena, M. Quiros and C.E.M. Wagner, "Effective potential methods and the Higgs mass spectrum in the MSSM," Page 418, Copyright (1996), with permission from Elsevier*

here to determine the limits from the 3 CMS analyses. In addition, the program `CheckMATE`[13] was used to evaluate bounds on the gluino mass for each model. `CheckMATE` requires as input a `HepMC` [209] file containing generated events and the production cross section of the sparticles of interest along with the total 1σ uncertainty on the cross section. We use `PYTHIA 8.175` [210] to generate 10,000–20,000 events in `HepMC` format.[14] The gluino production cross section and its uncertainty are obtained from [211]. It was found that the ATLAS analysis ATLAS-CONF-2013-061 [212], on which we elaborate below, is the most constraining analysis for each of the benchmark points in the models.

The ATLAS-CONF-2013-061 analysis is a search for final states with large missing transverse momentum, at least four, six, or seven jets, at least three jets tagged as b-jets, and either zero or at least one lepton. It was performed at $\sqrt{s} = 8\,\text{TeV}$ with 20.1 fb^{-1} of data. The results are interpreted in the context of a variety of simplified models. The simplified models considered that are relevant to the benchmark models are the Gbb, Gtt, and Gtb models. These models assume 100% branching ratios of a gluino to $b\bar{b}\tilde{\chi}_1^0$, $t\bar{t}\tilde{\chi}_1^0$, and $tb\tilde{\chi}_1^{\pm}$, respectively. The most constraining signal regions on the benchmark models are presented in Table 10.4. They are constrained most by the signal region requiring at least 1 lepton and at least 6 jets. Each of the benchmark points in these cases has a sizeable gluino branching fraction into $tb\tilde{\chi}_1^{\pm}$. The mass splitting of the lightest chargino and the lightest neutralino is large enough for the chargino to decay to a W whose decay

[13] `CheckMATE` uses `Delphes 3` [205], `FastJet` [206, 207], and the Anti-kt jet algorithm [208].

[14] We are using `HepMC 2.06.09`.

Table 10.4 Most constraining signal region for the universal and dark matter scenarios

Baseline selection: ≥ 1 signal lepton (e, μ), $p_T^{j_1} > 90\,\text{GeV}$, $E_T^{\text{miss}} > 150\,\text{GeV}$, ≥ 4 jets with $p_T > 30\,\text{GeV}$, ≥ 3 b-jets with $p_T > 30\,\text{GeV}$					
Signal region	N jets	E_T^{miss} (GeV)	m_T (GeV)	$m_{\text{eff}}^{\text{incl}}$ (GeV)	$E_T^{\text{miss}}/\sqrt{H_T^{\text{incl}}}$ (GeV$^{\frac{1}{2}}$)
SR-1l-6J-B	≥ 6	>225	>140	>800	>5

A detailed description of the parameters in this table can be found in [212]

products are energetic enough to be seen in the detector. We found that the lower bound on $M_{\tilde{g}}$ in Yukawa-unified SO(10) SUSY GUTs is generically $\sim$1.2 TeV at the 1σ level.

Thus we find that the third family analysis with universal or non-universal gaugino masses with $\alpha \leq 2.5$ is consistent with low energy data.

1. In order to fit low energy data we are forced to a restricted region of soft SUSY breaking parameters.
2. We therefore predict a well defined spectrum of SUSY particle masses.
3. In particular the gluino is expected to have mass in the range 1.2 TeV $\lesssim M_{\tilde{g}} \lesssim$ 2 TeV. (See the sentence preceding footnote 12 for more details.)

10.3 $SO(10)$ GUT with $D_3 \times [U(1) \times Z_2 \times Z_3]$ Family Symmetry

A complete model for fermion masses was given in [146, 190]. Using a global χ^2 analysis, it has been shown that the model fits all fermion masses and mixing angles, including neutrinos, and a minimal set of precision electroweak observables. The model is consistent with lepton flavor violation and lepton electric dipole moment bounds. In several recent papers, [170, 193, 213–215], the model was also tested by flavor violating processes in the B system.

The model is an $SO(10)$ SUSY GUT with an additional $D_3 \times [U(1) \times Z_2 \times Z_3]$ family symmetry.[15] The three families of quarks and leptons are contained in three 16 dimensional representations of $SO(10)$ $\{16_a,\ 16_3\}$ with $16_a,\ a = 1,2$ a D_3 flavor doublet (see Sect. 10.4 for details on D_3). The two MSSM Higgs doublets H_u and H_d are contained in a **10** where the Higgs, **10**, and 16_3 are both D_3 singlets. The symmetry group fixes the following structure for the superpotential

$$\mathscr{W} = \mathscr{W}_f + \mathscr{W}_\nu \,. \tag{10.11}$$

[15] For related charged fermion analyses in $SO(10)$ SUSY GUTS with $D_3 \times U(1)$ (or $SU(2) \times U(1)^n$) family symmetries, see [216] (or [139, 140, 217]). Note also that the groups D_3 and S_3 are isomorphic.

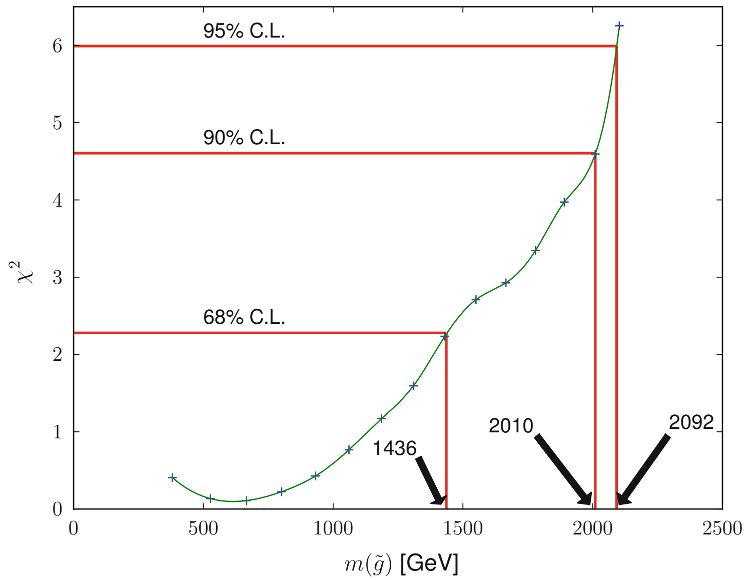

Fig. 10.3 The χ^2 function for fixed $m_{16} = 20\,\text{TeV}$ and different values of $M_{1/2}$ corresponding to the gluino masses indicated on the x-axis (figure taken from [168]). In minimizing χ^2 for each point, we followed the same procedure as in [213] except that we now do not penalize for gluino masses smaller than a certain lower bound. The confidence level intervals correspond to the χ^2 distribution with two degrees of freedom

The superpotential resulting in charged fermion masses and mixing angles is given by

$$\mathscr{W}_f = \lambda\, \mathbf{16}_3\, \mathbf{10}\, \mathbf{16}_3 + \mathbf{16}_a\, \mathbf{10}\, \chi_a \qquad (10.12)$$
$$+ \bar{\chi}_a (M_\chi\, \chi_a + \mathbf{45}\, \frac{\phi_a}{\hat{M}}\, \mathbf{16}_3 + \mathbf{45}\, \frac{\tilde{\phi}_a}{\hat{M}}\, \mathbf{16}_a + A\, \mathbf{16}_a)\ ,$$

where $M_\chi = M_0(1 + \alpha X + \beta Y)$ includes $SO(10)$ breaking VEVs in the X and Y directions, ϕ^a, $\tilde{\phi}^a$ (D_3 doublets), $\mathbf{A}$ ($\mathbf{1_B}$ singlet) are $SO(10)$ singlet flavon fields, and $\hat{M}$, M_0 are $SO(10)$ singlet masses. The fields 45, A, ϕ, $\tilde{\phi}$ are assumed to obtain VEVs $\langle 45 \rangle \sim (B - L)\, M_G$, $A \ll M_0$ and

$$\langle \phi \rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},\ \langle \tilde{\phi} \rangle = \begin{pmatrix} 0 \\ \tilde{\phi}_2 \end{pmatrix} \qquad (10.13)$$

with $\phi_1 > \phi_2$.

As can be seen from the first term on the right-hand side of (10.12), Yukawa unification $\lambda_t = \lambda_b = \lambda_\tau = \lambda_{\nu_\tau}$ at M_G is obtained *only* for the third generation, which is directly coupled to the Higgs $\mathbf{10}$ representation. This immediately implies large $\tan\beta \approx 50$ at low energies and constrains soft SUSY breaking parameters, as discussed earlier. The effective Yukawa couplings of the first and second generation fermions are generated hierarchically via the Froggatt-Nielsen mechanism [125]. The Froggatt-Nielsen states $\chi_a,\ \bar{\chi}_a$ transform as a $\mathbf{16}, \overline{\mathbf{16}}$, respectively and both are D_3 doublets. They receive mass of $O(M_G)$ as M_χ acquires an SO(10) breaking VEV. Once they are integrated out, they give rise to effective mass operators which, together with the VEVs of the flavon fields, create the Yukawa couplings for the first two generations. This mechanism breaks systematically the full flavor symmetry and produces the right mass hierarchies among the fermions.

The superpotential, [Eq. (10.12)] results in the following charged fermion Yukawa matrices[16]:

$$
\begin{aligned}
Y_u &= \begin{pmatrix} 0 & -\epsilon'\,\rho & \epsilon\,\xi \\ \epsilon'\,\rho & \tilde{\epsilon}\,\rho & \epsilon \\ -\epsilon\,\xi & -\epsilon & 1 \end{pmatrix} \lambda \\
Y_d &= \begin{pmatrix} 0 & -\epsilon' & \epsilon\,\xi \\ \epsilon' & \tilde{\epsilon} & \epsilon \\ -\epsilon\,\xi\,\sigma & -\epsilon\,\sigma & 1 \end{pmatrix} \lambda \\
Y_e &= \begin{pmatrix} 0 & \epsilon' & -3\,\epsilon\,\xi\,\sigma \\ -\epsilon' & 3\,\tilde{\epsilon} & -3\,\epsilon\,\sigma \\ 3\,\epsilon\,\xi & 3\,\epsilon & 1 \end{pmatrix} \lambda
\end{aligned}
\tag{10.14}
$$

with

$$
\begin{aligned}
\xi &= \phi_2/\phi_1; \quad \tilde{\epsilon} \propto \tilde{\phi}_2/\hat{M}; \\
\epsilon &\propto \phi_1/\hat{M}; \quad \epsilon' \sim (\mathbf{A}/M_0); \\
\sigma &= \frac{1+\alpha}{1-3\alpha}; \quad \rho \sim \beta \ll \alpha.
\end{aligned}
\tag{10.15}
$$

Let us now discuss neutrino masses. In the three 16s we have three electroweak doublet neutrinos $(\nu_a,\ \nu_3)$ and three electroweak singlet anti-neutrinos $(\bar{\nu}_a,\ \bar{\nu}_3)$.[17] The superpotential $\mathscr{W}_f$ also results in a neutrino Yukawa matrix:

$$
Y_\nu = \begin{pmatrix} 0 & \epsilon'\,\omega & -3\epsilon\,\xi\,\sigma \\ -\epsilon'\,\omega & 3\,\tilde{\epsilon}\,\omega & -3\,\epsilon\,\sigma \\ \frac{3}{2}\,\epsilon\,\xi\,\omega & \frac{3}{2}\,\epsilon\,\omega & 1 \end{pmatrix} \lambda
\tag{10.16}
$$

[16] It has been shown in [140] that excellent fits to charged fermion masses and mixing angles are obtained with this Yukawa structure.

[17] In an equivalent notation, we have three left-handed neutrinos $(\nu_{La} \equiv \nu_a,\ \nu_{L3} \equiv \nu_3)$ and three right-handed neutrinos defined by $(\nu_{Ra} \equiv \bar{\nu}_a^*,\ \nu_{R3} \equiv \bar{\nu}_3^*)$.

with $\omega = 2\sigma/(2\sigma - 1)$ and a Dirac neutrino mass matrix given by

$$m_\nu \equiv Y_\nu \frac{v}{\sqrt{2}} \sin\beta. \tag{10.17}$$

Note, that the theory is very predictive since, as a result of the GUT and family symmetries, the number of arbitrary parameters is restricted significantly.

In addition, the anti-neutrinos get GUT scale masses by mixing with three $SO(10)$ singlets $\{N_a,\ a = 1,2;\quad N_3\}$ transforming as a D_3 doublet and singlet respectively. The superpotential for the neutrino sector is given by

$$\begin{aligned} \mathscr{W}_\nu = \overline{16}\,(\lambda_2\, N_a\, 16_a \ +\ \lambda_3\, N_3\, 16_3) \\ +\ \frac{1}{2}\,(S_a\, N_a\, N_a \ +\ S_3\, N_3\, N_3)\,. \end{aligned} \tag{10.18}$$

We assume $\overline{16}$ obtains a vev, v_{16}, in the right-handed neutrino direction, and $\langle S_a \rangle = M_a$ for $a = 1,2$ (with $M_2 > M_1$) and $\langle S_3 \rangle = M_3$.[18]

We thus obtain the effective neutrino mass terms given by

$$\mathscr{W}_\nu^{\mathit{eff}} = \nu\, m_\nu\, \bar{\nu} + \bar{\nu}\, V\, N + \frac{1}{2}\, N\, M_N\, N \tag{10.19}$$

with

$$V = v_{16} \begin{pmatrix} 0 & \lambda_2 & 0 \\ \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},\ M_N = diag(M_1,\ M_2,\ M_3). \tag{10.20}$$

A simple family symmetry giving the desired form of the superpotential[19] is $D_3 \times U(1) \times Z_2 \times Z_3$ where the D_3 charges were defined earlier, while the $U(1)$ charge assignments are (1 for 16_3, 2 for 16_a, -2 for N_a, -1 for N_3, -1 for 45, 0 for $\overline{16}$ and $\overline{\chi}_a$) and everyone else fixed by these. In addition we assign Z_2 charges $(16_3, 16_a, N_3, N_a, \overline{\chi}_a, \chi_a)$ odd, all others even and Z_3 charges $\alpha = e^{\frac{2\pi i}{3}}$ for all fields, except 45 with charge 1. *Note, that Z_2 can also be interpreted as a family reflection symmetry which guarantees an unbroken low energy R parity* [87].

The electroweak singlet neutrinos $\{\bar{\nu}, N\}$ have large masses $V, M_N \sim M_G$. After integrating out these heavy neutrinos, we obtain the light neutrino mass matrix in the charged lepton flavor basis given by

$$\tilde{m}_\nu = U_l^T\, m_\nu^T\, M_R^{-1}\, m_\nu\, U_l = U_l^T\, \mathscr{M}\, U_l, \tag{10.21}$$

[18] These are the most general set of vevs for ϕ_a and S_a. The zero vev for $\tilde{\phi}_1$ can be enforced with a simple superpotential term such as $S\,\tilde{\phi}_a\,\tilde{\phi}_a$.

[19] Note, highly suppressed terms of the form $S_a\ N_a\ N_3\ \tilde{\phi}_a\ \phi_a^2$ are still allowed. This too can be forbidden by an additional discrete Z_4 symmetry.

where the effective right-handed neutrino Majorana mass matrix is given by:

$$M_R = V\, M_N^{-1}\, V^T \;\equiv\; \mathrm{diag}(M_{R_1}, M_{R_2}, M_{R_3}), \tag{10.22}$$

with

$$M_{R_1} = (\lambda_2\, v_{16})^2/M_2, \quad M_{R_2} = (\lambda_2\, v_{16})^2/M_1, \quad M_{R_3} = (\lambda_3\, v_{16})^2/M_3. \tag{10.23}$$

U_l is the 3×3 unitary matrix for left-handed leptons needed to diagonalize Y_e [Eq. (10.14)], i.e. $Y_e^D = U_{\bar{e}}^\dagger\, Y_e\, U_l$ and also U such that $U^T\, \tilde{m}_\nu\, U = \tilde{m}_\nu^D = \mathrm{diag}(m_{\nu_1},\ m_{\nu_2},\ m_{\nu_3})$, then the neutrino mixing matrix is given by $U = U_{PMNS}$ in terms of the flavor eigenstate $(\nu_\alpha, \alpha = e,\ \mu,\ \tau)$ and mass eigenstate $(\nu_i, i = 1, 2, 3)$ basis fields with

$$\nu_\alpha = \sum_i (U_{PMNS})_{\alpha i}\, \nu_i. \tag{10.24}$$

For U_{PMNS} we use the notation of [218] with

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{i\alpha_1/2}\nu_1 \\ e^{i\alpha_2/2}\nu_2 \\ \nu_3 \end{pmatrix} \tag{10.25}$$

The 3×3 Majorana mass matrix is of the general form discussed by many authors [219]. We have

$$\mathscr{M} = \mathscr{P}_1\, \mathscr{M}_1\, \mathscr{P}_1 + \mathscr{P}_2\, \mathscr{M}_2\, \mathscr{P}_2 + \mathscr{P}_3\, \mathscr{M}_3\, \mathscr{P}_3 \tag{10.26}$$

with

$$\mathscr{M}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\epsilon'\, |\omega|)^2 & -3\,\epsilon'\,\epsilon\, |\xi\, \sigma\, \omega| \\ 0 & -3\,\epsilon'\,\epsilon\, |\xi\, \sigma\, \omega| & (3\,\epsilon\, |\xi\, \sigma|)^2 \end{pmatrix} \left(\frac{\lambda\ v\ \sin\beta}{\sqrt{2}\, \lambda_2\, v_{16}} \right)^2 M_2; \tag{10.27}$$

$$\mathscr{M}_2 = \begin{pmatrix} (\epsilon'\, |\omega|)^2 & -3\,\epsilon'\, |\tilde{\epsilon}\, \omega^2| & 3\,\epsilon'\,\epsilon\, |\sigma\, \omega| \\ -3\,\epsilon'\, |\tilde{\epsilon}\, \omega^2| & (3\, |\tilde{\epsilon}\, \omega|)^2 & -9\,\epsilon\, |\tilde{\epsilon}\, \sigma\, \omega| \\ 3\,\epsilon'\,\epsilon\, |\sigma\, \omega| & -9\,\epsilon\, |\tilde{\epsilon}\, \sigma\, \omega| & (3\,\epsilon\, |\sigma|)^2 \end{pmatrix} \left(\frac{\lambda\ v\ \sin\beta}{\sqrt{2}\, \lambda_2\, v_{16}} \right)^2 M_1; \tag{10.28}$$

$$\mathscr{M}_3 = \begin{pmatrix} (\frac{3}{2}\,\epsilon\, |\xi\, \omega|)^2 & |\xi|\, (\frac{3}{2}\,\epsilon\, |\omega|)^2 & \frac{3}{2}\,\epsilon\, |\xi\, \omega| \\ |\xi|\, (\frac{3}{2}\,\epsilon\, |\omega|)^2 & (\frac{3}{2}\,\epsilon\, |\omega|)^2 & \frac{3}{2}\,\epsilon\, |\omega| \\ \frac{3}{2}\,\epsilon\, |\xi\, \omega| & \frac{3}{2}\,\epsilon\, |\omega| & 1 \end{pmatrix} \left(\frac{\lambda\ v\ \sin\beta}{\sqrt{2}\, \lambda_3\, v_{16}} \right)^2 M_3 \tag{10.29}$$

and

$$\mathscr{P}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\Phi_\omega} & 0 \\ 0 & 0 & e^{-i(\Phi_\sigma + \Phi_\xi)} \end{pmatrix}; \tag{10.30}$$

$$\mathscr{P}_2 = \begin{pmatrix} e^{-i\Phi_\omega} & 0 & 0 \\ 0 & e^{-i(\Phi_{\tilde{\epsilon}} + \Phi_\omega)} & 0 \\ 0 & 0 & e^{-i\Phi_\sigma} \end{pmatrix}; \tag{10.31}$$

$$\mathscr{P}_3 = \begin{pmatrix} e^{-i(\Phi_\omega + \Phi_\xi)} & 0 & 0 \\ 0 & e^{-i\Phi_\omega} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10.32}$$

$\mathscr{M}_1$, $\mathscr{M}_2$, $\mathscr{M}_3$ are in general complex rank 1 mass matrices. However only the difference in their overall phases may be observable. Thus, there are, in principle, two new CP violating phases in the neutrino sector, in addition to the four phases already fixed by charged fermion masses and mixing angles. *We shall impose the constraint that neutrino Majorana masses M_i are all real.* This eliminates two arbitrary phases. We note that the best fits with free phases for M_i are very close numerically to our zero phase model.

10.3.1 Full Three Family Global χ^2 Analysis

Procedure

Here we just explain the additional steps in the procedure. There are now 24 (or 25 with α) parameters in the model (see Table 10.5). First, the GUT scale parameters

Table 10.5 The model has 24 free parameters for universal gaugino masses boundary conditions

Sector	Universal gaugino masses	No.	Mirage mediated gaugino masses	No.
Gauge	$\alpha_G, M_G, \epsilon_3$	3	$\alpha_G, M_G, \epsilon_3$	3
SUSY (GUT scale)	$m_{16}, M_{1/2}, A_0, m_{H_u}, m_{H_d}$	5	$m_{16}, M_{1/2}, A_0, m_{H_u}, m_{H_d}, \alpha$	6
Yukawa textures	$\epsilon, \epsilon', \lambda, \rho, \sigma, \tilde{\epsilon}, \xi, \phi_\rho, \phi_\sigma, \phi_{\tilde{\epsilon}}, \phi_\xi$	11	$\epsilon, \epsilon', \lambda, \rho, \sigma, \tilde{\epsilon}, \xi, \phi_\rho, \phi_\sigma, \phi_{\tilde{\epsilon}}, \phi_\xi$	11
Neutrino	$M_{R_1}, M_{R_2}, M_{R_3}$	3	$M_{R_1}, M_{R_2}, M_{R_3}$	3
SUSY (EW scale)	$\tan\beta, \mu$	2	$\tan\beta, \mu$	2
Total		24		25

For mirage mediated gaugino masses boundary condition, there is an additional input parameter, α, which determines the amount of splitting of the gaugino masses at GUT scale are RGE evolved to the right-handed neutrino scale where the RH neutrinos are integrated out. The right-handed neutrinos have three different scales associated with them, and the most relevant one is the third-family RHN that is mostly responsible for splitting the up and down type Higgs masses. We therefore choose to integrate out all the right-handed neutrinos at one single scale, $M_{N_\tau} = M_{R_3}$.

Charged Fermion Masses and Mixing Angles

Below M_Z, we integrate out all SUSY partners and electroweak gauge bosons to obtain an effective $SU(3) \times U(1)_{EM}$ low energy theory. We use 1-loop QED and 3-loop QCD RGEs to renormalize to the appropriate scales and calculate the low energy observables. We fit the top quark pole mass, and the bottom and charm quark $\overline{\text{MS}}$ masses are calculated at their respective masses. All the other light quark $\overline{\text{MS}}$ masses are calculated at the scale of 2 GeV. We fit seven observables relevant to quark masses, three charged lepton masses, and six CKM observables. The theoretical uncertainty in their calculation is again estimated to be 0.5%. Since the light quark masses are not measured to very high precision, we choose to fit multiple correlated observables. These include the $\overline{\text{MS}}$ strange quark mass, the mass ratio m_d/m_s and the mass ratio Q defined in the PDG [220] as

$$Q^2 = \frac{m_s^2 - 1/4(m_u + m_d)^2}{m_d^2 - m_u^2}, \quad \text{or equivalently,} \quad \left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1 \tag{10.33}$$

The CKM matrix is calculated from the left and right mixing matrices by diagonalizing the Yukawa matrices and including the SUSY threshold corrections. Six CKM observables ($|V_{us}|$, $|V_{ub}|$, $|V_{cb}|$, $|V_{td}|$, $|V_{ts}|$ and $\sin 2\beta$) are included in our global fit analysis. To account for the inconsistencies in the inclusive and exclusive measurements of $|V_{ub}|$ and $|V_{cb}|$, we allow our result to be within the experimental error from both the inclusive and the exclusive measurement. We also separately fit either the inclusive or exclusive values of $|V_{ub}|$ and $|V_{cb}|$. The pole masses in the lepton sector are calculated with 1-loop electromagnetic threshold corrections.

To execute the steps elaborated so far, we use a code `maton` , originally developed by Radovan Dermíšek to study Yukawa unification in the SO(10) model with $D_3 \times [U1 \times \mathbb{Z}_2 \times \mathbb{Z}_3]$ family symmetry [190]. `maton` has been restructured and extended appropriately (by Archana Anandakrishnan, B. Charles Bryant, Zijie Poh, and Akin Wingerter) to adapt to the current analysis.

Neutrino Sector

We are fitting five observables in the neutrino sector: the mixing angles θ_{12}, θ_{23}, θ_{13}, and the mass-squared differences $\Delta m_{31} \equiv m_3^2 - m_1^2$ and $\Delta m_{21} \equiv m_2^2 - m_1^2$. The most dramatic change in the experimental determination of the neutrino parameters in recent years comes from the Daya Bay and Reno collaborations [221, 222] that have confirmed that $\theta_{13} \sim 9°$ is indeed large. Moreover, there are tentative hints that θ_{23} is not maximal [223, 224]. Whereas [223] sees a preference at $\sim 2\sigma - 3\sigma$ for the first octant, i.e. $\theta_{23} < 45°$, [224] finds an equal probability for θ_{23} being larger or smaller than 45°. In the following, we use the best-fit values and the 3σ uncertainties quoted by the NuFIT collaboration [224] which are in agreement with [223] at 3σ.

Flavor Physics

The strongest constraints on the model come from B-physics. For calculating the flavor observables, we use two publicly available codes, namely susy_flavor [225] and SuperIso [226, 227]. Since the boundary conditions that we impose at the GUT scale may generate large off-diagonal and in general complex entries at the low scale, susy_flavor is better adapted to our needs. Note that susy_flavor, in contrast to comparable programs that calculate similar processes, does not assume minimal flavor violation (MFV), and allows for general, full three family, complex soft parameters. This is particularly important in our case, since we are calculating several CP violating observables and need to take into account[20] the complex phases in the soft parameters. Hence, susy_flavor is our default choice for all flavor observables with the following exceptions. For $B \to X_s\gamma$, we use SuperIso, since susy_flavor does not include the NNLO SM corrections. We have verified that the discrepancy between susy_flavor and SuperIso in the parameter space that is of interest to us is at most 10% and typically less than 7%. Also, we use SuperIso for the observables connected to the decay process $B \to K^*\mu^+\mu^-$, since susy_flavor does not provide them. It is important to note that SuperIso has some built-in assumptions that prove to be too restrictive in our case. E.g. SuperIso assumes all soft parameters to be real, and only takes the diagonal entries of the third-family trilinear couplings into account. As a consequence, we have assigned larger theoretical uncertainties to the values calculated by SuperIso. Additional sources of uncertainties in the flavor observables derive from the theoretical determination of the B meson decay constant and from the experimental measurements of the CKM matrix elements.

LHCb has recently measured the $\mathrm{Br}(B_s \to \mu^+\mu^-)$ [228] which is in good agreement with the SM prediction. This pushes the CP-odd Higgs mass to a few TeV and hence leads to the Higgs decoupling limit. *Thus the light Higgs is predicted to be SM-like.* The recent observation of zero-crossing in the forward-backward asymmetry of $B \to K^*\mu^+\mu^-$ constrains the Wilson coefficient C_7 to be of the same

[20] We calculate the particle spectrum using maton, see comments on p. 104. To the best of our knowledge, there is currently no publicly available spectrum generator that fully takes into account all the complex phases of the MSSM.

sign as that in the SM. This imposes the additional constraint for the model that if $\mu > 0$, in order to satisfy the branching fraction observed in the process $B \to X_s\gamma$ the first two generation scalars have to be heavier than at least 10 TeV.

Global Fit

In the last step we define a χ^2 function and fit the 45 observables in Table 10.6 (see [170]).

10.3.2 Results: Global χ^2 Analysis

Several benchmark points with the results of the global χ^2 analysis are given in Tables 10.9, 10.10, 10.11, 10.12 and a plot of χ^2 as a function of the parameter m_{16} is given in Fig. 10.4 or χ^2 contours in the two dimensional plane of $M_{\tilde{g}}$ vs. m_{16} is given in Fig. 10.6. Let us now discuss some features of the results.

Inclusive vs. Exclusive $|V_{ub}|$ and $|V_{cb}|$

Due to the discrepancy between the values of $|V_{ub}|$ and $|V_{cb}|$ determined from inclusive and exclusive semi-leptonic decay. We define three different χ^2 functions:

1. $|V_{ub}|$ and $|V_{cb}|$ are taken to be the inclusive values
2. $|V_{ub}|$ and $|V_{cb}|$ are taken to be the exclusive values
3. $|V_{ub}|$ and $|V_{cb}|$ are taken to be the average of inclusive and exclusive values with error bars overlapping with the error bars of both the inclusive and the exclusive measurements

The results of these three analyses are shown in Fig. 10.4. We see that for both the universal boundary condition $\alpha = 0$ and mirage boundary condition with $\alpha = 1.5$, the χ^2/d.o.f. obtained by fitting to the inclusive values are the biggest. Hence, we predict that the exclusive values of $|V_{ub}|$ and $|V_{cb}|$ are the correct values for both universal and mirage gaugino masses.

Since the χ^2 difference between case (2) and case (3) is small and to be conservative, the analyses of the rest of the paper are done for case (3), where $|V_{ub}|$ and $|V_{cb}|$ are the average of the inclusive and exclusive values.

Table 10.6 Forty-five observables that we fit

Observable	Exp. value	Ref.	Program	Th. error
M_Z	91.1876 ± 0.0021 GeV	[47]	Input	0.0%
M_W	80.385 ± 0.015 GeV	[47]	maton	0.5%
α_{em}	1/137.035999074(44)	[47]	maton	0.5%
G_μ	$1.1663787(6) \times 10^{-5}$ GeV^{-2}	[47]	maton	1%
$\alpha_3(M_Z)$	0.1185 ± 0.0006	[47]	maton	0.5%
M_t	$173.21 \pm 0.51 \pm 0.71$ GeV	[47]	maton	0.5%
$m_b(m_b)$	4.18 ± 0.03 GeV	[47]	maton	0.5%
M_τ	1776.82 ± 0.16 MeV	[47]	maton	0.5%
$m_b - m_c$	3.45 ± 0.05 GeV	[47]	maton	10%
$m_c(m_c)$	1.275 ± 0.025 GeV	[47]	maton	0.5%
m_s(2 GeV)	95 ± 5 MeV	[47]	maton	0.5%
m_s/m_d (2 GeV)	17–22	[47]	maton	0.5%
Q	21–25	[47]	maton	5%
M_μ	105.6583715(35) MeV	[47]	maton	0.5%
M_e	0.510998928(11) MeV	[47]	maton	0.5%
$\lvert V_{us}\rvert$	0.2253 ± 0.0008	[47]	maton	0.5%
$\lvert V_{cb}\rvert$ (Inclusive)	0.0422 ± 0.0007	[47]	maton	0.5%
$\lvert V_{cb}\rvert$ (Exclusive)	0.0395 ± 0.0008	[47]	maton	0.5%
$\lvert V_{cb}\rvert$ (Both)	0.0408 ± 0.0021	[47]	maton	0.5%
$\lvert V_{ub}\rvert$ (Inclusive)	0.00441 ± 0.00024	[47]	maton	0.5%
$\lvert V_{ub}\rvert$ (Exclusive)	0.00328 ± 0.00029	[47]	maton	0.5%
$\lvert V_{ub}\rvert$ (Both)	0.00385 ± 0.00086	[47]	maton	0.5%
$\lvert V_{td}\rvert$	0.00840 ± 0.0006	[47]	maton	0.5%
$\lvert V_{ts}\rvert$	0.0400 ± 0.0027	[47]	maton	0.5%
$\sin 2\beta$	0.682 ± 0.019	[47]	maton	0.5%
ϵ_K	$(2.2325 \pm 0.0155) \times 10^{-3}$	[47]	susyflavor[225]	10%
$\Delta m_{B_s}/\Delta m_{B_d}$	35.0345 ± 0.3884	[47]	susyflavor[225]	20%
Δm_{B_d}	$(3.337 \pm 0.033) \times 10^{-10}$ MeV	[47]	susyflavor[225]	20%
Δm^2_{21}	(7.02–8.09) $\times 10^{-5}$ eV^2 (3σ range)	[229]	maton	0.5%
Δm^2_{31}	(2.317–2.607) $\times 10^{-3}$ eV^2 (3σ range)	[229]	maton	0.5%
$\sin^2\theta_{12}$	0.270–0.344 (3σ range)	[229]	maton	0.5%
$\sin^2\theta_{23}$	0.382–0.643 (3σ range)	[229]	maton	0.5%
$\sin^2\theta_{13}$	0.0186–0.0250 (3σ range)	[229]	maton	0.5%
M_h	125.7 ± 0.4 GeV	[47]	splitsuspect[187]	3 GeV
BR($b \to s\gamma$)	$(343 \pm 21 \pm 7) \times 10^{-6}$	[230]	superiso[227]	40%
BR($B_s \to \mu^+\mu^-$)	$(2.8^{+0.7}_{-0.6}) \times 10^{-9}$	[231]	susyflavor[225]	20%

(continued)

Table 10.6 (continued)

$\mathrm{BR}(B_d \to \mu^+\mu^-)$	$(3.9^{+1.6}_{-1.4}) \times 10^{-10}$	[231]	`susyflavor`[225]	20%
$\mathrm{BR}(B \to \tau\nu)$	$(114 \pm 22) \times 10^{-6}$	[230]	`susyflavor`[225]	50%
$\mathrm{BR}(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\mathrm{GeV}^2}$	$0.34 \pm 0.03 \pm 0.04 \pm 0.02^{+0.00}_{-0.03} \times 10^{-7}$	[232]	`superiso`[227]	105%
$\mathrm{BR}(B \to K^*\mu^+\mu^-)_{14.18\le q^2\le 16\,\mathrm{GeV}^2}$	$0.45 \pm 0.06 \pm 0.04 \pm 0.04^{+0.00}_{-0.05} \times 10^{-7}$	[232]	`superiso`[227]	190%
$q_0^2(\mathrm{A_{FB}}(B \to K^*\mu^+\mu^-))$	$4.9 \pm 0.9\,\mathrm{GeV}^2$	[232]	`superiso`[227]	25%
$F_L(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\mathrm{GeV}^2}$	$0.65^{+0.08+0.03}_{-0.07-0.03}$	[232]	`superiso`[227]	45%
$F_L(B \to K^*\mu^+\mu^-)_{14.18\le q^2\le 16\,\mathrm{GeV}^2}$	$0.33^{+0.08+0.03}_{-0.07-0.03}$	[232]	`superiso`[227]	80%
$-2P_2 = A_T^{\mathrm{Re}}(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\mathrm{GeV}^2}$	$-0.66^{+0.24+0.04}_{-0.22-0.01}$	[232]	`superiso`[227]	95%
$-2P_2 = A_T^{\mathrm{Re}}(B \to K^*\mu^+\mu^-)_{14.18\le q^2\le 16\,\mathrm{GeV}^2}$	$1.00^{+0.00+0.00}_{-0.05-0.02}$	[232]	`superiso`[227]	45%
$P_4'(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\mathrm{GeV}^2}$	$0.58^{+0.36}_{-0.32} \pm 0.06$	[233]	`superiso`[227]	30%
$P_4'(B \to K^*\mu^+\mu^-)_{14.18\le q^2\le 16\,\mathrm{GeV}^2}$	$-0.18^{+0.70}_{-0.54} \pm 0.08$	[233]	`superiso`[227]	35%
$P_5'(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\mathrm{GeV}^2}$	$0.21^{+0.20}_{-0.21} \pm 0.03$	[233]	`superiso`[227]	45%
$P_5'(B \to K^*\mu^+\mu^-)_{14.18\le q^2\le 16\,\mathrm{GeV}^2}$	$-0.79^{+0.20}_{-0.13} \pm 0.18$	[233]	`superiso`[227]	60%

All experimental errors are 1σ unless otherwise indicated. Column 4 shows the software package that gives us the theoretical prediction. M_Z is fit precisely to impose electroweak symmetry breaking. To account for the inconsistencies in the inclusive and exclusive measurements of $|V_{ub}|$ and $|V_{cb}|$, we perform a global χ^2 analysis using the inclusive and exclusive measurement separately. We also perform an additional analysis where the error bars of $|V_{ub}|$ and $|V_{cb}|$ cover both the inclusive and exclusive values. In addition to the error bars indicated in the `superiso` manual, we added an extra 15% error to $B \to K^*\mu^+\mu^-$ observables because `superiso` does not take into account the phases of soft terms

10.3.3 SUSY Non-decoupled Observables

B Physics Observables

Some of the measured angular observables of $B \to K^*\mu^+\mu^-$ are in tension with the SM prediction. For example, P_4' in the high q^2 bin ($14.18 \le q^2 \le 16\,\mathrm{GeV}^2$) has a 2.7σ discrepancy with the SM prediction, P_5' in the low q^2 bin ($1 \le q^2 \le 6\,\mathrm{GeV}^2$) has a 2.5σ discrepancy with the SM prediction, and P_2 in the low q^2 bin has a 2σ discrepancy with the SM prediction [232–234]. In addition, Altmannshofer et al. [234] found the tension in P_4' of the high q^2 bin cannot be explained by the MSSM. On the other hand, the tension of F_L and P_5' of the low q^2 bin can be explained by the MSSM by having a negative contribution to the C_7 Wilson coefficient. In the

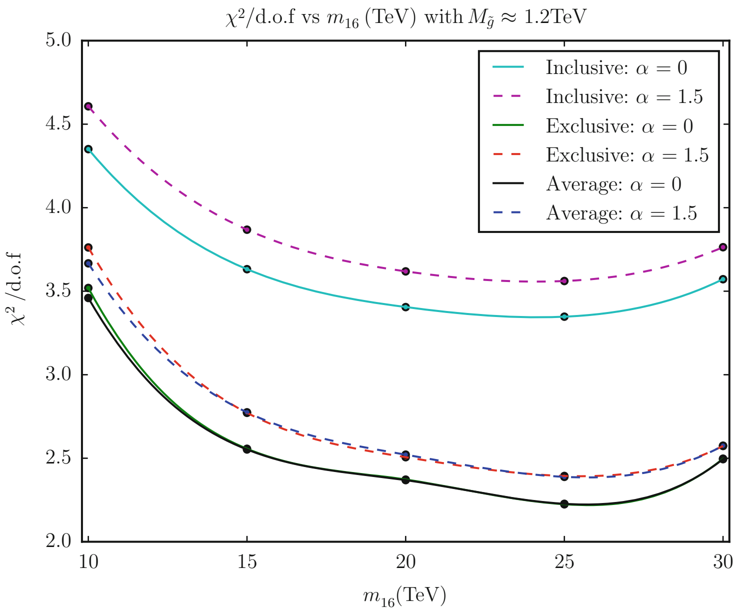

Fig. 10.4 This plot shows the value of χ^2/d.o.f. as a function of m_{16} for cases where the value of $|V_{ub}|$ and $|V_{cb}|$ are taken to be the inclusive values, the exclusive values, or the average of inclusive and exclusive values. *Solid lines* refers to the universal boundary condition, $\alpha = 0$, while *dashed lines* refer to the mirage boundary condition with $\alpha = 1.5$. This plot shows that our model favors the exclusive values of $|V_{ub}|$ and $|V_{cb}|$

standard model $C_7 \approx -0.32$. The tension in F_L and P'_5 can be further reduced by making C_7 more negative [234].

In the MSSM, chargino-stop loops and charged Higgs loop contribute to C_7. The C_7 contribution from the charged Higgs is always negative. The charged Higgs of our model has mass around 2 TeV. So, the charged Higgs contribution to C_7 is non-negligible and is in the correct direction.

The chargino-stop loop contribution of C_7^{MSSM} has the following form [214][21]

$$C_7^{\text{MSSM}} = \frac{\mu A_t \tan\beta}{m_{\tilde{t}}^4} \text{sign}(C_7^{\text{SM}}) \,. \tag{10.34}$$

Since sign(μA_t) is negative in our model, this term contributes to C_7 in the wrong direction. Hence, to reduce the contribution of this term, our model favors large scalar masses.

[21] Note, the equation for the chargino contribution to C_7^{MSSM} given in Eq. (21), [234] apparently has the wrong sign.

From our global χ^2 analysis, we see that the calculated value of P_4' in the high q^2 bin does not depend on m_{16}, which is expected. In addition, the value of P_4' calculated in our model is in agreement with the SM. Hence, our results are in agreement with Altmannshofer et al.'s claim that the tension in P_4' cannot be explained in the MSSM. As shown in Table 10.7, the tension of F_L and P_5' with the experimental values decreases as m_{16} increases. This is again in agreement with our expectation as explained above.

SUSY Corrections to the W Mass

On the other hand, the correction for M_W is given by [235–237]

$$\delta M_W \approx \frac{M_W}{2}\frac{c_W^2}{c_W^2 - s_W^2}\Delta\rho \tag{10.35}$$

and the 1-loop squark contribution is given by

$$\Delta\rho_1^{\mathrm{SUSY}} = \frac{3G_\mu}{8\sqrt{2}\pi^2}[-s_{\tilde{t}}^2 c_{\tilde{t}}^2 F_0(m_{\tilde{t}_1}^2, m_{\tilde{t}_2}^2) - s_{\tilde{b}}^2 c_{\tilde{b}}^2 F_0(m_{\tilde{b}_1}^2, m_{\tilde{b}_2}^2) \tag{10.36}$$

$$+ c_{\tilde{t}}^2 c_{\tilde{b}}^2 F_0(m_{\tilde{t}_1}^2, m_{\tilde{b}_1}^2) + c_{\tilde{t}}^2 s_{\tilde{b}}^2 F_0(m_{\tilde{t}_1}^2, m_{\tilde{b}_2}^2)$$

$$+ s_{\tilde{t}}^2 c_{\tilde{b}^2} F_0(m_{\tilde{t}_2}^2, m_{\tilde{b}_1}^2) + s_{\tilde{t}}^2 s_{\tilde{b}}^2 F_0(m_{\tilde{t}_2}^2, m_{\tilde{b}_2^2})] \tag{10.37}$$

where $s_W = \sin\theta_W, c_W = \cos\theta_W, s_{\tilde{q}} = \sin\theta_{\tilde{q}}, c_{\tilde{q}} = \cos\theta_{\tilde{q}}$, and

$$F_0(x, y) = x + y - \frac{2xy}{x-y}\ln\frac{x}{y}\,. \tag{10.38}$$

F_0 has properties of $F_0(x, x) = 0$ and $F_0(x, 0) = x$. Hence, we see that when the mass splitting of the squarks is large, the SUSY contribution to the 1-loop M_W can be significant. This is in agreement with our analysis which shows that the pull from M_W increases as the value of m_{16} increases above 20 TeV. Hence, SUSY corrections to M_W are significant and they can go in the right direction.

Light Higgs Mass

Fitting to the Higgs mass also constraints the value of m_{16}. The dominant one-loop contribution to the Higgs mass is given by

$$m_h^2 \approx m_Z^2\cos^2 2\beta + \frac{3}{(4\pi)^2}\frac{m_t^4}{v^2}\left[\ln\frac{M_{SUSY}^2}{m_t^2} + \frac{X_t}{M_{SUSY}^2}\left(1 - \frac{X_t^2}{12M_{SUSY}^2}\right)\right] \tag{10.39}$$

Table 10.7 This table shows the set of observables with $\alpha = 0$ and $M_{\tilde{g}} \approx 1.2\,\text{TeV}$

	Pull				
m_{16}	10	15	20	25	30
M_W	0.2110	0.1878	0.1851	0.2320	0.3981
M_h	2.5474	1.1795	0.3454	0.1882	0.6582
$BR(B \to \tau\nu)$	1.1978	1.3952	1.3557	1.3588	1.3771
$F_L(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\text{GeV}^2}$	0.2696	0.2488	0.2219	0.2101	0.2057
$P_4'(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\text{GeV}^2}$	1.7066	1.7066	1.7066	1.7066	1.7066
$P_5'(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\text{GeV}^2}$	2.4110	2.3432	2.2746	2.2451	2.2339
χ^2	14.1511	8.9744	7.2154	7.0206	7.5220

	Fit value					Exp.
m_{16}	10	15	20	25	30	value
M_W	80.4699	80.4606	80.4595	80.4784	80.5454	80.3850
M_h	117.9901	122.1303	124.6547	126.2697	127.6920	125.7000
$BR(B \to \tau\nu) \times 10^5$	6.6329	6.1340	6.2299	6.2223	6.1778	11.4000
$F_L(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\text{GeV}^2}$	0.7434	0.7353	0.7251	0.7207	0.7191	0.6500
$P_4'(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\text{GeV}^2}$	0.8174	0.6711	0.5921	0.5717	0.5657	0.5800
$P_4'(B \to K^*\mu^+\mu^-)_{14.18\le q^2\le 16\,\text{GeV}^2}$	1.2190	1.2190	1.2190	1.2190	1.2190	−0.1800
$P_5'(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\text{GeV}^2}$	−0.7301	−0.5529	−0.4625	−0.4335	−0.4235	0.2100

As argued by Altmannshofer et al. $B \to K^*\mu^+\mu^-$ favors large $m_{\tilde{t}}$ [234], which is in agreement with our analysis. On the other hand, as explained in the text, fitting to M_W and M_h disfavors large $m_{\tilde{t}}$. These effects collectively contribute to having a minimum χ^2 around $m_{16} = 25\,\text{TeV}$. The χ^2 value is plotted as a function of m_{16} in Fig. 10.5. In addition to the observables contributing directly to a minimum of χ^2 at $m_{16} = 25\,\text{TeV}$, also included in this table is the calculated value of P_4' at high q^2 bin. This is to illustrate that the value of P_4' does not depend on the m_{16}, which again agrees with Altmannshofer et al. [234]

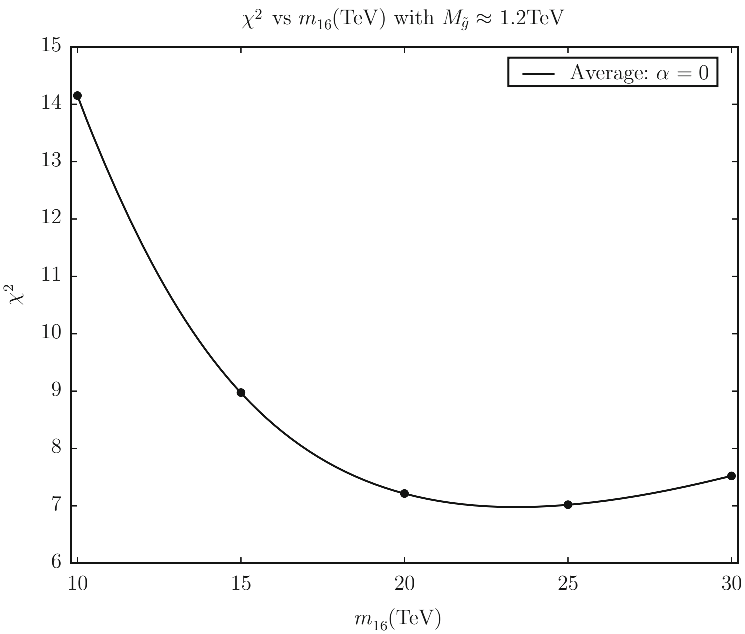

Fig. 10.5 This figure shows the contribution to χ^2 as a function of m_{16} just from the set of observables listed in Table 10.7. Fitting to the values of M_W, M_h, and b-physics observables listed in Table 10.7 helps explain why χ^2 is minimized at $m_{16} \approx 25$ TeV

where $X_t = A_t - \mu/\tan\beta$ is the stop mixing parameter and $M^2_{SUSY} = m_{\tilde{t}_1} \times m_{\tilde{t}_2}$. In our model, $X_t < -\sqrt{6}\, M_{SUSY}$ and the ratio X_t/M_{SUSY} becomes less negative as m_{16} increases. Hence, as m_{16} increases, the Higgs mass also increases (see Fig. 10.2). The pull in χ^2 due to M_h has a minimum around $m_{16} = 25$ TeV.

Summary

Hence, the contributions of M_W, M_h, and b-physics observables to χ^2, as listed in Table 10.7, helps explain the shape of χ^2 as a function of m_{16} (see Fig. 10.5).

10.3.4 Bounds on $M_{\tilde{g}}$

To obtain a better picture for the favored value of gluino mass, we plotted two contour plots of $M_{\tilde{g}}$ vs. m_{16}. One for the universal boundary condition $\alpha = 0$ and another for the mirage boundary condition with $\alpha = 1.5$. The current lower bound on $M_{\tilde{g}}$, for our model, is around 1.2 TeV [167]. The contour plots are created by calculating χ^2/d.o.f. for 25 equally distributed values of m_{16} and $M_{1/2}$, which gives us $1\,\text{TeV} < M_{\tilde{g}} < 2\,\text{TeV}$. We then use cubic interpolation to obtain the smooth contours of χ^2/d.o.f.

In addition to the contour lines of the χ^2/d.o.f., we also plotted a 4σ contour line. From this, we see that for mirage boundary conditions $M_{\tilde{g}} \lesssim 1.8\,\text{TeV}$. However, for universal boundary condition, the 4σ $M_{\tilde{g}}$ bound can be as high as 3 TeV, which is not shown in the figure.[22] Hence, for mirage boundary conditions, we expect the 4σ bound on the gluino mass to be within reach in the next run of the LHC. As pointed out by [167], the dominant decay mode of the gluino in the universal gaugino mass boundary condition is $t\ b\ \tilde{\chi}_1^{\mp}$. The remaining decay modes are $t\ \bar{t}\ \tilde{\chi}_i^0$ and $b\ \bar{b}\ \chi_i^0$ for $i = 1, 2$. On the other hand, the dominant mode for gluino decay in the mirage gaugino mass boundary condition is $t\ b\ \tilde{\chi}_i^{\mp}$ for $i = 1, 2$. In all cases, the dominant signature for gluinos in this model is given by b jets, leptons and missing E_T [167].

10.3.5 Additional Predictions for Some Benchmark Points

The mass spectrum of the benchmark point of $M_{\tilde{g}} \approx 1.2\,\text{TeV}$ and $m_{16} = 25\,\text{TeV}$ is shown in Table 10.8. From the χ^2 analysis, we see that the scalar masses are predicted to be around 5 TeV, while the first and second generation scalars have mass around $m_{16} \approx 25\,\text{TeV}$. With scalars in this mass range, the stop in our model does not completely decouple and can have non-negligible effects on flavor physics. In addition, our light Higgs is SM-like with the heavy Higgs with mass around 2 TeV.

In Table 10.8, we give the light sparticle masses, the CP violating angle for neutrino oscillations, δ, the branching ratio $BR(\mu \to e\gamma)$ and the electric dipole moment of the electron for two different values of $M_{\tilde{g}}$ and for $\alpha = 0$ and 1.5. Note that, in general, the gauginos are the lightest sparticles. In addition, $BR(\mu \to e\gamma)$ and the electric dipole moment of the electron are within reach of future experiments.

10.3.6 Conclusion: SUSY on the Edge

We have analyzed a three family SO(10) SUSY GUT with Yukawa unification for the third family. The model gives reasonable fits to fermion masses and mixing angles, as well as many other low energy observables; see Tables 10.9, 10.10, 10.11, 10.12 with some benchmark points of the global χ^2 analysis. A plot of χ^2 as a function of the parameter m_{16} is given in Fig. 10.4 or χ^2 contours in the two dimensional plane of $M_{\tilde{g}}$ vs. m_{16} is given in Fig. 10.6.

[22] These bounds are consistent with our third family analysis.

Table 10.8 Predictions with $m_{16} = 25\,\text{TeV}$ for $M_{\tilde{g}} \approx 1.2$ and 1.6 TeV

m_{16}	25	25	25	25
α	0	1.5	0	1.5
χ^2/d.o.f.	2.158	2.275	2.220	2.505
$m_{\tilde{t}_1}$	4.903	5.011	4.909	5.249
$m_{\tilde{t}_2}$	6.021	6.120	6.033	6.301
$m_{\tilde{b}_1}$	5.989	6.088	6.455	6.606
$m_{\tilde{b}_2}$	6.454	6.541	6.445	6.267
$m_{\tilde{\tau}_1}$	9.880	9.931	9.912	10.040
$m_{\tilde{\tau}_2}$	15.369	15.365	15.393	15.516
$M_{\tilde{g}}$	1.202	1.187	1.613	1.690
$m_{\tilde{\chi}_1^0}$	0.203	0.551	0.279	0.900
$m_{\tilde{\chi}_2^0}$	0.404	0.665	0.538	1.018
$m_{\tilde{\chi}_1^+}$	0.404	0.665	0.538	1.018
$m_{\tilde{\chi}_2^+}$	1.128	1.243	1.232	1.537
M_A	2.194	2.082	2.477	3.352
$\sin\delta$	−0.289	−0.482	−0.520	−0.576
$BR(\mu \to e\gamma) \times 10^{13}$	1.108	1.430	1.239	1.340
$\text{edm}_e \times 10^{30}\,(\text{e cm})$	−1.403	−3.305	−1.763	−5.886

All masses in the table are in TeV units. Our prediction for the branching ratio $\mu \to e\gamma$ is consistent with the current upper bound of 5.7×10^{-13} [47]. In addition, our prediction of the electron electric dipole moment is consistent with the current upper bound of 10.5×10^{-28} e cm [47]

We performed an analysis with universal gaugino masses and with non-universal gaugino mass with splitting determined by "mirage mediation" boundary conditions described in Eq. (10.5). The parameter $\alpha = 0$ for universal gaugino masses and we also take $\alpha = 1.5$ which is consistent with a well-tempered dark matter candidate [238]. In both cases the model favors $m_{16} \approx 25\,\text{TeV}$. Nevertheless, due to RG running [184], stops and sbottoms have mass of order 5 TeV, while the first two family scalar masses are of order m_{16}. With m_{16} lying in this mass range, stops in our model do not completely decouple from low energy flavor observables (see Sect. 10.3.3). Best fits are found with a gluino mass less than 2 TeV. Our gluinos decay predominantly into third generation quarks [167]. Moreover, in a previous analysis [167] we showed that the dominant LHC signature for gluinos in the model is given by b-Jets, leptons and missing E_T. Note that, in general, the gauginos are the lightest sparticles. The CP odd Higgs mass is of order 2 TeV, thus the light Higgs couplings are very much Standard Model-like. In Table 10.8 we present additional predictions. We give the predictions for the CP violating angle for neutrino oscillations, δ, the branching ratio $BR(\mu \to e\gamma)$ and the electric dipole moment of the electron for two different values of $M_{\tilde{g}}$ and for $\alpha = 0$ and 1.5. In addition, $BR(\mu \to e\gamma)$ and the electric dipole moment of the electron are within reach of future experiments. Thus this theory is eminently testable!

Table 10.9 Benchmark point with $m_{16} = 25\,\text{TeV}, M_{\tilde{g}} = 1.202\,\text{TeV}, \alpha = 0$:
$(1/\alpha_G, M_G, \epsilon_3) = (25.98, 2.55 \times 10^{16}\,\text{GeV}, -1.30\%)$
$(\lambda, \lambda\epsilon, \sigma, \lambda\tilde{\epsilon}, \rho, \lambda\epsilon', \lambda\epsilon\xi) = (0.6101, 0.0308, 1.1559, 0.0049, 0.0698, -0.0019, 0.0036)$
$(\phi_\sigma, \phi_{\tilde{\epsilon}}, \phi_\rho, \phi_\xi) = (0.52, 0.58, 3.95, 3.47)$ rad
$(m_{16}, M_{1/2}, A_0, \mu(M_Z)) = (25000, 280, -51380, 1212)$ GeV
$((m_{H_d}/m_{16})^2, (m_{H_u}/m_{16})^2, \tan\beta) = (1.86, 1.61, 50.29)$
$(M_{R_1}, M_{R_2}, M_{R_3}) = (9.2, 578.8, 35054.2) \times 10^9$ GeV

Observable	Fit	Exp.	Pull	σ
M_Z	91.1876	91.1876	0.0000	0.4541
M_W	80.4784	80.3850	0.2320	0.4027
$1/\alpha_{\text{em}}$	137.2810	0.0073	0.3569	0.6864
$G_\mu \times 10^5$	1.1789	1.1664	1.0598	0.0118
$\alpha_3(M_Z)$	0.1192	0.1185	0.8199	0.0008
M_t	174.0947	173.2100	0.7171	1.2337
$m_b(m_b)$	4.1986	4.1800	0.5092	0.0366
m_τ	1.7772	1.7768	0.0428	0.0089
$M_b - M_c$	3.1701	3.4500	0.8720	0.3209
$m_c(m_c)$	1.2509	1.2750	0.9333	0.0258
$m_s(2\,\text{GeV})$	0.0953	0.0950	0.0609	0.0050
$m_d/m_s(2\,\text{GeV})$	0.0702	0.0513	2.8247	0.0067
$1/Q^2$	0.0018	0.0019	0.4528	0.0001
M_μ	0.1056	0.1057	0.0578	0.0005
$M_e \times 10^4$	5.1143	5.1100	0.1674	0.0256
$\lvert V_{us}\rvert$	0.2245	0.2253	0.5931	0.0014
$\lvert V_{cb}\rvert$	0.0404	0.0408	0.1670	0.0021
$\lvert V_{ub}\rvert \times 10^3$	3.1235	3.8500	0.8446	0.8601
$\lvert V_{td}\rvert \times 10^3$	8.8463	8.4000	0.7418	0.6016
$\lvert V_{ts}\rvert$	0.0396	0.0400	0.1508	0.0027
$\sin 2\beta$	0.6296	0.6820	2.7214	0.0193
ϵ_K	0.0022	0.0022	0.0022	0.0002
$\Delta M_{B_s}/\Delta M_{B_d}$	34.8195	35.0345	0.0308	6.9747
$\Delta M_{B_d} \times 10^{13}$	3.9946	3.3370	0.8224	0.7996
$m_{21}^2 \times 10^5$	7.5883	7.5550	0.0621	0.5363
$m_{31}^2 \times 10^3$	2.4649	2.4620	0.0197	0.1455
$\sin^2\theta_{12}$	0.3028	0.3070	0.1125	0.0370
$\sin^2\theta_{23}$	0.6600	0.5125	1.1300	0.1305
$\sin^2\theta_{13}$	0.0162	0.0218	1.7510	0.0032
M_h	126.2697	125.7000	0.1882	3.0265
$BR(B \to s\gamma) \times 10^4$	2.7220	3.4300	0.5419	1.3064
$BR(B_s \to \mu^+\mu^-) \times 10^9$	2.7213	2.8000	0.0888	0.8867
$BR(B_d \to \mu^+\mu^-) \times 10^{10}$	1.0734	3.9000	1.7509	1.6143
$BR(B \to \tau\nu) \times 10^5$	6.2223	11.4000	1.3588	3.8104
$BR(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\text{GeV}^2} \times 10^8$	4.7860	3.4000	0.2739	5.0610

(continued)

Table 10.9 (continued)

Observable	Fit	Exp.	Pull	σ
$BR(B \to K^* \mu^+ \mu^-)_{14.18 \le q^2 \le 16\,\mathrm{GeV}^2} \times 10^8$	7.5495	5.6000	0.1356	14.3788
$q_0^2(A_{\mathrm{FB}}(B \to K^* \mu^+ \mu^-))$	3.7120	4.9000	0.9190	1.2927
$F_L(B \to K^* \mu^+ \mu^-)_{1 \le q^2 \le 6\,\mathrm{GeV}^2}$	0.7207	0.6500	0.2101	0.3366
$F_L(B \to K^* \mu^+ \mu^-)_{14.18 \le q^2 \le 16\,\mathrm{GeV}^2}$	0.3108	0.3300	0.0726	0.2644
$P_2(B \to K^* \mu^+ \mu^-)_{1 \le q^2 \le 6\,\mathrm{GeV}^2}$	0.0331	0.3300	2.3939	0.1240
$P_2(B \to K^* \mu^+ \mu^-)_{14.18 \le q^2 \le 16\,\mathrm{GeV}^2}$	−0.4336	−0.5000	0.3364	0.1974
$P_4'(B \to K^* \mu^+ \mu^-)_{1 \le q^2 \le 6\,\mathrm{GeV}^2}$	0.5717	0.5800	0.0208	0.3988
$P_4'(B \to K^* \mu^+ \mu^-)_{14.18 \le q^2 \le 16\,\mathrm{GeV}^2}$	1.2190	−0.1800	1.7066	0.8198
$P_5'(B \to K^* \mu^+ \mu^-)_{1 \le q^2 \le 6\,\mathrm{GeV}^2}$	−0.4335	0.2100	2.2451	0.2866
$P_5'(B \to K^* \mu^+ \mu^-)_{14.18 \le q^2 \le 16\,\mathrm{GeV}^2}$	−0.7116	−0.7900	0.1552	0.5052
Total χ^2			46.7692	

Table 10.10 Benchmark point with $m_{16} = 25\,\mathrm{TeV}, M_{\tilde{g}} = 1.187\,\mathrm{TeV}, \alpha = 1.5$:
$(1/\alpha_G, M_G, \epsilon_3) = (25.95, 2.60 \times 10^{16}\,\mathrm{GeV}, -1.50\%)$
$(\lambda, \lambda\epsilon, \sigma, \lambda\tilde{\epsilon}, \rho, \lambda\epsilon', \lambda\epsilon\xi) = (0.6100, 0.0310, 1.1459, 0.0049, 0.0708, -0.0019, 0.0037)$
$(\phi_\sigma, \phi_{\tilde{\epsilon}}, \phi_\rho, \phi_\xi) = (0.53, 0.57, 3.94, 3.49)$ rad
$(m_{16}, M_{1/2}, A_0, \mu(M_Z)) = (25000, 520, -51157, 1236)$ GeV
$((m_{H_d}/m_{16})^2, (m_{H_u}/m_{16})^2, \tan\beta) = (1.85, 1.61, 50.19)$
$(M_{R_1}, M_{R_2}, M_{R_3}) = (9.1, 567.0, 32370.5) \times 10^9$ GeV

Observable	Fit	Exp.	Pull	σ
M_Z	91.1876	91.1876	0.0000	0.4540
M_W	80.5197	80.3850	0.3344	0.4029
$1/\alpha_{\mathrm{em}}$	137.1416	0.0073	0.1540	0.6857
$G_\mu \times 10^5$	1.1829	1.1664	1.3978	0.0118
$\alpha_3(M_Z)$	0.1189	0.1185	0.4798	0.0008
M_t	173.8449	173.2100	0.5150	1.2328
$m_b(m_b)$	4.2023	4.1800	0.6094	0.0366
m_τ	1.7772	1.7768	0.0450	0.0089
$M_b - M_c$	3.1680	3.4500	0.8791	0.3207
$m_c(m_c)$	1.2570	1.2750	0.6979	0.0258
$m_s(2\,\mathrm{GeV})$	0.0947	0.0950	0.0671	0.0050
$m_d/m_s(2\,\mathrm{GeV})$	0.0700	0.0513	2.7901	0.0067
$1/Q^2$	0.0018	0.0019	0.5027	0.0001
M_μ	0.1056	0.1057	0.1457	0.0005
$M_e \times 10^4$	5.1145	5.1100	0.1775	0.0256
$\lvert V_{us}\rvert$	0.2244	0.2253	0.6440	0.0014
$\lvert V_{cb}\rvert$	0.0407	0.0408	0.0584	0.0021
$\lvert V_{ub}\rvert \times 10^3$	3.1307	3.8500	0.8363	0.8601
$\lvert V_{td}\rvert \times 10^3$	8.8596	8.4000	0.7639	0.6016
$\lvert V_{ts}\rvert$	0.0398	0.0400	0.0652	0.0027

(continued)

Table 10.10 (continued)

$\sin 2\beta$	0.6285	0.6820	2.7790	0.0193
ϵ_K	0.0023	0.0022	0.1149	0.0002
$\Delta M_{B_s}/\Delta M_{B_d}$	35.5946	35.0345	0.0786	7.1295
$\Delta M_{B_d} \times 10^{13}$	3.9756	3.3370	0.8025	0.7958
$m_{21}^2 \times 10^5$	7.6111	7.5550	0.1046	0.5364
$m_{31}^2 \times 10^3$	2.4657	2.4620	0.0255	0.1455
$\sin^2 \theta_{12}$	0.3134	0.3070	0.1724	0.0370
$\sin^2 \theta_{23}$	0.6319	0.5125	0.9146	0.1305
$\sin^2 \theta_{13}$	0.0153	0.0218	2.0337	0.0032
M_h	124.5455	125.7000	0.3814	3.0265
$BR(B \to s\gamma) \times 10^4$	2.7270	3.4300	0.5372	1.3087
$BR(B_s \to \mu^+\mu^-) \times 10^9$	2.5215	2.8000	0.3228	0.8627
$BR(B_d \to \mu^+\mu^-) \times 10^{10}$	1.0192	3.9000	1.7861	1.6129
$BR(B \to \tau\nu) \times 10^5$	6.2272	11.4000	1.3568	3.8124
$BR(B \to K^*\mu^+\mu^-)_{1\leq q^2\leq 6\,\mathrm{GeV}^2} \times 10^8$	4.8580	3.4000	0.2839	5.1361
$BR(B \to K^*\mu^+\mu^-)_{14.18\leq q^2\leq 16\,\mathrm{GeV}^2} \times 10^8$	7.6648	5.6000	0.1415	14.5975
$q_0^2(A_{\mathrm{FB}}(B \to K^*\mu^+\mu^-))$	3.7150	4.9000	0.9163	1.2933
$F_L(B \to K^*\mu^+\mu^-)_{1\leq q^2\leq 6\,\mathrm{GeV}^2}$	0.7208	0.6500	0.2103	0.3366
$F_L(B \to K^*\mu^+\mu^-)_{14.18\leq q^2\leq 16\,\mathrm{GeV}^2}$	0.3108	0.3300	0.0726	0.2644
$P_2(B \to K^*\mu^+\mu^-)_{1\leq q^2\leq 6\,\mathrm{GeV}^2}$	0.0335	0.3300	2.3879	0.1242
$P_2(B \to K^*\mu^+\mu^-)_{14.18\leq q^2\leq 16\,\mathrm{GeV}^2}$	−0.4336	−0.5000	0.3364	0.1974
$P_4'(B \to K^*\mu^+\mu^-)_{1\leq q^2\leq 6\,\mathrm{GeV}^2}$	0.5697	0.5800	0.0258	0.3985
$P_4'(B \to K^*\mu^+\mu^-)_{14.18\leq q^2\leq 16\,\mathrm{GeV}^2}$	1.2190	−0.1800	1.7066	0.8198
$P_5'(B \to K^*\mu^+\mu^-)_{1\leq q^2\leq 6\,\mathrm{GeV}^2}$	−0.4334	0.2100	2.2450	0.2866
$P_5'(B \to K^*\mu^+\mu^-)_{14.18\leq q^2\leq 16\,\mathrm{GeV}^2}$	−0.7117	−0.7900	0.1550	0.5052
Total χ^2			47.7692	

Table 10.11 Benchmark point with $m_{16} = 25\,\mathrm{TeV}, M_{\tilde{g}} = 1.613\,\mathrm{TeV}, \alpha = 0$:
$(1/\alpha_G, M_G, \epsilon_3) = (26.22, 2.32 \times 10^{16}\,\mathrm{GeV}, -0.65\%)$
$(\lambda, \lambda\epsilon, \sigma, \lambda\tilde{\epsilon}, \rho, \lambda\epsilon', \lambda\epsilon\xi) = (0.6096, 0.0311, 1.1398, 0.0049, 0.0710, -0.0019, 0.0038)$
$(\phi_\sigma, \phi_{\tilde{\epsilon}}, \phi_\rho, \phi_\xi) = (0.53, 0.56, 3.95, 3.49)$ rad
$(m_{16}, M_{1/2}, A_0, \mu(M_Z)) = (25, 450, -51341, 1226)\,\mathrm{GeV}$
$((m_{H_d}/m_{16})^2, (m_{H_u}/m_{16})^2, \tan\beta) = (1.86, 1.61, 50.30)$
$(M_{R_1}, M_{R_2}, M_{R_3}) = (9.1, 572.4, 32277.4) \times 10^9\,\mathrm{GeV}$

Observable	Fit	Exp.	Pull	σ
M_Z	91.1876	91.1876	0.0000	0.4535
M_W	80.4507	80.3850	0.1633	0.4025
$1/\alpha_{\mathrm{em}}$	137.7125	0.0073	0.9825	0.6886
$G_\mu \times 10^5$	1.1732	1.1664	0.5798	0.0117
$\alpha_3(M_Z)$	0.1188	0.1185	0.4140	0.0008

(continued)

Table 10.11 (continued)

M_t	174.1882	173.2100	0.7927	1.2340
$m_b(m_b)$	4.1954	4.1800	0.4220	0.0366
m_τ	1.7781	1.7768	0.1417	0.0089
$M_b - M_c$	3.1568	3.4500	0.9175	0.3196
$m_c(m_c)$	1.2595	1.2750	0.5993	0.0258
$m_s(2\,\text{GeV})$	0.0939	0.0950	0.2147	0.0050
$m_d/m_s(2\,\text{GeV})$	0.0701	0.0513	2.8052	0.0067
$1/Q^2$	0.0018	0.0019	0.5139	0.0001
M_μ	0.1056	0.1057	0.1818	0.0005
$M_e \times 10^4$	5.1145	5.1100	0.1749	0.0256
$\|V_{us}\|$	0.2244	0.2253	0.6763	0.0014
$\|V_{cb}\|$	0.0404	0.0408	0.1729	0.0021
$\|V_{ub}\| \times 10^3$	3.1033	3.8500	0.8681	0.8601
$\|V_{td}\| \times 10^3$	8.8101	8.4000	0.6817	0.6016
$\|V_{ts}\|$	0.0396	0.0400	0.1531	0.0027
$\sin 2\beta$	0.6270	0.6820	2.8562	0.0193
ϵ_K	0.0022	0.0022	0.2052	0.0002
$\Delta M_{B_s}/\Delta M_{B_d}$	35.3739	35.0345	0.0479	7.0854
$\Delta M_{B_d} \times 10^{13}$	3.9433	3.3370	0.7681	0.7894
$m_{21}^2 \times 10^5$	7.6562	7.5550	0.1886	0.5364
$m_{31}^2 \times 10^3$	2.4631	2.4620	0.0077	0.1455
$\sin^2\theta_{12}$	0.3170	0.3070	0.2689	0.0370
$\sin^2\theta_{23}$	0.6264	0.5125	0.8722	0.1305
$\sin^2\theta_{13}$	0.0149	0.0218	2.1658	0.0032
M_h	124.5054	125.7000	0.3947	3.0265
$BR(B \to s\gamma) \times 10^4$	2.6840	3.4300	0.5789	1.2887
$BR(B_s \to \mu^+\mu^-) \times 10^9$	3.0247	2.8000	0.2429	0.9252
$BR(B_d \to \mu^+\mu^-) \times 10^{10}$	1.1022	3.9000	1.7323	1.6151
$BR(B \to \tau\nu) \times 10^5$	6.1884	11.4000	1.3727	3.7966
$BR(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\text{GeV}^2} \times 10^8$	4.7640	3.4000	0.2707	5.0381
$BR(B \to K^*\mu^+\mu^-)_{14.18\le q^2\le 16\,\text{GeV}^2} \times 10^8$	7.5110	5.6000	0.1336	14.3059
$q_0^2(A_{\text{FB}}(B \to K^*\mu^+\mu^-))$	3.6690	4.9000	0.9579	1.2850
$F_L(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\text{GeV}^2}$	0.7225	0.6500	0.2149	0.3374
$F_L(B \to K^*\mu^+\mu^-)_{14.18\le q^2\le 16\,\text{GeV}^2}$	0.3108	0.3300	0.0726	0.2644
$P_2(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\text{GeV}^2}$	0.0228	0.3300	2.5196	0.1219
$P_2(B \to K^*\mu^+\mu^-)_{14.18\le q^2\le 16\,\text{GeV}^2}$	−0.4336	−0.5000	0.3364	0.1974
$P_4'(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\text{GeV}^2}$	0.5820	0.5800	0.0050	0.4001
$P_4'(B \to K^*\mu^+\mu^-)_{14.18\le q^2\le 16\,\text{GeV}^2}$	1.2190	−0.1800	1.7066	0.8198
$P_5'(B \to K^*\mu^+\mu^-)_{1\le q^2\le 6\,\text{GeV}^2}$	−0.4455	0.2100	2.2578	0.2903
$P_5'(B \to K^*\mu^+\mu^-)_{14.18\le q^2\le 16\,\text{GeV}^2}$	−0.7116	−0.7900	0.1552	0.5052
Total χ^2			48.8413	

Table 10.12 Benchmark point with $m_{16} = 25\,\text{TeV}, M_{\tilde{g}} = 1.690\,\text{TeV}, \alpha = 1.5$:
$(1/\alpha_G, M_G, \epsilon_3) = (26.38, 2.09 \times 10^{16}\,\text{GeV}, 0.02\%)$
$(\lambda, \lambda\epsilon, \sigma, \lambda\tilde{\epsilon}, \rho, \lambda\epsilon', \lambda\epsilon\xi) = (0.6096, 0.0311, 1.1384, 0.0049, 0.0708, -0.0019, 0.0037)$
$(\phi_\sigma, \phi_{\tilde{\epsilon}}, \phi_\rho, \phi_\xi) = (0.52, 0.56, 3.96, 3.49)$ rad
$(m_{16}, M_{1/2}, A_0, \mu(M_Z)) = (25, 900, -50846, 1529)$ GeV
$((m_{H_d}/m_{16})^2, (m_{H_u}/m_{16})^2, \tan\beta) = (1.86, 1.60, 50.31)$
$(M_{R_1}, M_{R_2}, M_{R_3}) = (9.1, 579.0, 32367.3) \times 10^9$ GeV

Observable	Fit	Exp.	Pull	σ
M_Z	91.1876	91.1876	0.0000	0.4540
M_W	80.4655	80.3850	0.2000	0.4026
$1/\alpha_{\text{em}}$	137.7323	0.0073	1.0111	0.6887
$G_\mu \times 10^5$	1.1740	1.1664	0.6469	0.0117
$\alpha_3(M_Z)$	0.1188	0.1185	0.2979	0.0008
M_t	174.3427	173.2100	0.9175	1.2345
$m_b(m_b)$	4.2001	4.1800	0.5479	0.0366
m_τ	1.7774	1.7768	0.0644	0.0089
$M_b - M_c$	3.1659	3.4500	0.8863	0.3205
$m_c(m_c)$	1.2574	1.2750	0.6825	0.0258
$m_s(2\,\text{GeV})$	0.0936	0.0950	0.2741	0.0050
$m_d/m_s(2\,\text{GeV})$	0.0701	0.0513	2.8082	0.0067
$1/Q^2$	0.0018	0.0019	0.5170	0.0001
M_μ	0.1056	0.1057	0.1571	0.0005
$M_e \times 10^4$	5.1139	5.1100	0.1545	0.0256
$\|V_{us}\|$	0.2244	0.2253	0.6688	0.0014
$\|V_{cb}\|$	0.0400	0.0408	0.3609	0.0021
$\|V_{ub}\| \times 10^3$	3.0662	3.8500	0.9113	0.8601
$\|V_{td}\| \times 10^3$	8.7156	8.4000	0.5247	0.6016
$\|V_{ts}\|$	0.0392	0.0400	0.2960	0.0027
$\sin 2\beta$	0.6259	0.6820	2.9122	0.0193
ϵ_K	0.0022	0.0022	0.0834	0.0002
$\Delta M_{B_s}/\Delta M_{B_d}$	34.7964	35.0345	0.0342	6.9701
$\Delta M_{B_d} \times 10^{13}$	3.8958	3.3370	0.7165	0.7799
$m_{21}^2 \times 10^5$	7.6614	7.5550	0.1984	0.5364
$m_{31}^2 \times 10^3$	2.4606	2.4620	0.0094	0.1455
$\sin^2\theta_{12}$	0.3197	0.3070	0.3423	0.0370
$\sin^2\theta_{23}$	0.6197	0.5125	0.8210	0.1305
$\sin^2\theta_{13}$	0.0146	0.0218	2.2520	0.0032
M_h	122.0502	125.7000	1.2059	3.0265
$BR(B \to s\gamma) \times 10^4$	2.6310	3.4300	0.6321	1.2640
$BR(B_s \to \mu^+\mu^-) \times 10^9$	3.5145	2.8000	0.7203	0.9920
$BR(B_d \to \mu^+\mu^-) \times 10^{10}$	1.0522	3.9000	1.7647	1.6138

(continued)

Table 10.12 (continued)

$BR(B \to \tau\nu) \times 10^5$	6.1009	11.4000	1.4090	3.7610
$BR(B \to K^*\mu^+\mu^-)_{1 \le q^2 \le 6\,\text{GeV}^2} \times 10^8$	4.6780	3.4000	0.2583	4.9484
$BR(B \to K^*\mu^+\mu^-)_{14.18 \le q^2 \le 16\,\text{GeV}^2} \times 10^8$	7.4066	5.6000	0.1281	14.1080
$q_0^2(A_{\text{FB}}(B \to K^*\mu^+\mu^-))$	3.6290	4.9000	0.9946	1.2779
$F_L(B \to K^*\mu^+\mu^-)_{1 \le q^2 \le 6\,\text{GeV}^2}$	0.7240	0.6500	0.2189	0.3380
$F_L(B \to K^*\mu^+\mu^-)_{14.18 \le q^2 \le 16\,\text{GeV}^2}$	0.3108	0.3300	0.0726	0.2644
$P_2(B \to K^*\mu^+\mu^-)_{1 \le q^2 \le 6\,\text{GeV}^2}$	0.0132	0.3300	2.6254	0.1207
$P_2(B \to K^*\mu^+\mu^-)_{14.18 \le q^2 \le 16\,\text{GeV}^2}$	−0.4337	−0.5000	0.3358	0.1975
$P_4'(B \to K^*\mu^+\mu^-)_{1 \le q^2 \le 6\,\text{GeV}^2}$	0.5918	0.5800	0.0294	0.4014
$P_4'(B \to K^*\mu^+\mu^-)_{14.18 \le q^2 \le 16\,\text{GeV}^2}$	1.2190	−0.1800	1.7066	0.8198
$P_5'(B \to K^*\mu^+\mu^-)_{1 \le q^2 \le 6\,\text{GeV}^2}$	−0.4562	0.2100	2.2685	0.2937
$P_5'(B \to K^*\mu^+\mu^-)_{14.18 \le q^2 \le 16\,\text{GeV}^2}$	−0.7117	−0.7900	0.1550	0.5052
Total χ^2			52.6056	

Finally, with the large value of $m_{16} \sim 25\,\text{TeV}$ we expect the gravitino mass to be at least this large. Perhaps it is large enough to avoid a cosmological gravitino problem [239]. In addition, moduli may also be suitably heavy to avoid a cosmological moduli problem [240–243]. Hence the scalar masses are clearly in an intermediate range, i.e. too heavy to be "natural," But lighter than "Split SUSY." We thus are positioned on the border between these two limiting cases, i.e. this is "SUSY on the **Edge**."

10.4 The Group D_3 and Its Representations

In Sect. 10.3 we considered the $SO(10)$ model with a D_3 family symmetry. In this chapter we derive the representations and product rules for this family symmetry. All possible rotations in three dimensions which leave an equilateral triangle invariant form the group D_3 (see Fig., 10.7) (for more information see the Appendix of [216]). This group contains six elements in three classes[23]:

$$E; \quad C_3,\, C_3^2; \quad C_a,\, C_b,\, C_c,$$

where E is the identity element, C_3 is the rotation through $2\pi/3$ about the axis perpendicular to the paper and going through the center of the triangle, C_3^2 is C_3

[23] An element b of the group G is said to be *conjugate* to the element a if there is an element u in G such that $uau^{-1} = b$. A group can be separated into *classes* of elements which are conjugate to one another.

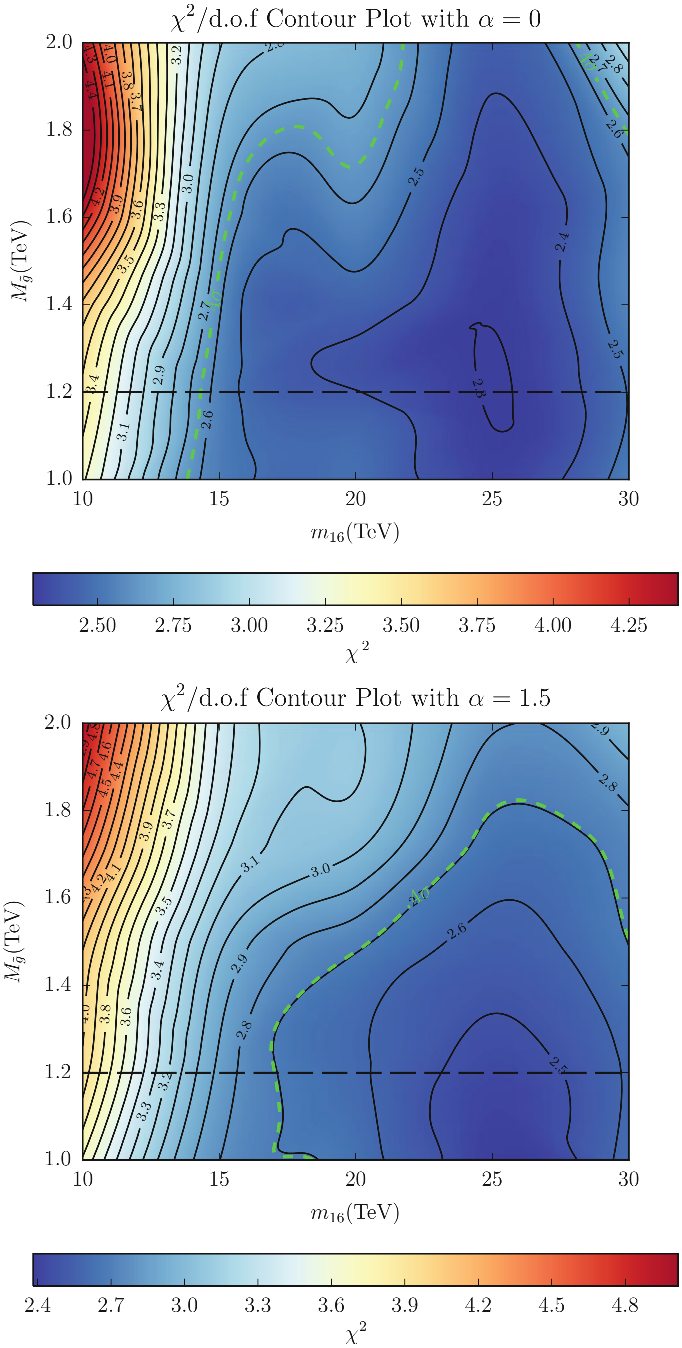

Fig. 10.6 These plots show the contour of χ^2/d.o.f. as a function of the $M_{\tilde{g}}$ and m_{16}. The 4σ bound is also included in the plot. For $\alpha = 1.5$, the upper bound is within reach of the next run of LHC. In addition, we also see that our model favors $m_{16} \approx 25$ TeV

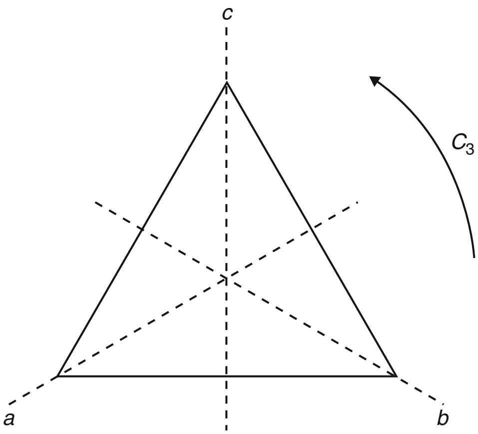

Fig. 10.7 Symmetry axes of an equilateral triangle

Table 10.13 The character table for the group D_3

D_3	E	C_3	C_a
$\mathbf{1_A}$	1	1	1
$\mathbf{1_B}$	1	1	−1
$\mathbf{2_A}$	2	−1	0

applied twice, C_a is the rotation through π about the axis a, and similarly C_b and C_c. Note that C_b is the same as C_aC_3 and C_c is the same as $C_aC_3^2$. The number of classes in a finite group is equal to the number of nonequivalent irreducible representations of the group. One of the most interesting results of the theory of finite groups is the relation between the number of elements g of a group and dimensions n_ν of its nonequivalent irreducible representations ν,

$$\sum_\nu n_\nu^2 = g.$$

Thus we find that the group D_3 has two nonequivalent one dimensional representations $\mathbf{1_A}$, $\mathbf{1_B}$ and one two dimensional representation $\mathbf{2_A}$. Each representation is described by the set of characters[24] $\chi_1, \ldots, \chi_\nu$, where ν is the number of classes in the group. The character table for the group D_3 is given in Table 10.13.

From the character table it is possible to find the decomposition of the product of any two representations:

$$\mathbf{1_A} \otimes \mathbf{1_A} = \mathbf{1_A}, \quad \mathbf{1_A} \otimes \mathbf{1_B} = \mathbf{1_B}, \quad \mathbf{1_B} \otimes \mathbf{1_B} = \mathbf{1_A}, \tag{10.40}$$

$$\mathbf{1_A} \otimes \mathbf{2_A} = \mathbf{2_A}, \quad \mathbf{1_B} \otimes \mathbf{2_A} = \mathbf{2_A}, \tag{10.41}$$

$$\mathbf{2_A} \otimes \mathbf{2_A} = \mathbf{1_A} \oplus \mathbf{1_B} \oplus \mathbf{2_A}. \tag{10.42}$$

[24]The character of an element a of the group G in a given representation D is the trace $\sum_i D_{ii}(a)$. Therefore elements in the same class (conjugate elements) have the same character.

To construct an explicit model obeying D_3 symmetry we need to specify the representation and determine invariant tensors. One dimensional representations coincide with the characters and the two dimensional representation can be chosen to be:

$$D(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad D(C_3) = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} , \quad D(C_a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{10.43}$$

where $\epsilon = e^{2\pi i/3}$.

Now it is straightforward to find the two singlets and the doublet in the decomposition of a product of two doublets (10.42). Writing $\psi = \{x, y\}$ and $\psi' = \{x', y'\}$, we find:

$$\psi \otimes \psi' \,|_{1_A} = xy' + yx' , \tag{10.44}$$

$$\psi \otimes \psi' \,|_{1_B} = xy' - yx' , \tag{10.45}$$

$$\psi \otimes \psi' \,|_2 = \begin{pmatrix} yy' \\ xx' \end{pmatrix} . \tag{10.46}$$

The decomposition (10.42) also reveals that the product of three doublets contains an invariant. Taking $\psi'' = \{x'', y''\}$, this invariant is:

$$\psi \otimes \psi' \otimes \psi'' \,|_{1_A} = xx'x'' + yy'y'' . \tag{10.47}$$

Finally, we want to show that given a doublet ψ_a in D_3, there is a unique invariant norm given by $\psi_a^* \psi_a \equiv \psi_1^* \psi_1 + \psi_2^* \psi_2$. Clearly, this norm is D_3 invariant since under a D_3 transformation $\psi_a' = C_{ab} \psi_b$ with $C \subset D_3$ and $C^\dagger C = 1$. That this is unique follows from the fact that in the product of two doublets there is a unique invariant given in Eq. (10.44). In addition, defining a new doublet by $\chi_a = g_{ab} \psi_b^*$ satisfying $\chi_a' = C_{ab} \chi_b = (\psi_b^*)' g_{ba}^T = \psi_c^* C_{cb}^\dagger g_{ba}^T$ requires for consistency $g = C g C^T$. The unique solution to this consistency condition is $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then we have $\chi \otimes \psi \,|_{1_A} \equiv \psi_a^* \psi_a$.

Chapter 11
Baryogenesis, Cosmological Moduli and Gravitino Problems, and Dark Matter

Supersymmetric theories have many applications to the early universe and cosmology. The Lightest Supersymmetric Particle, in a theory with R parity, is absolutely stable and can make an excellent dark matter candidate. In addition, cosmology places significant constraints on some superparticle masses. In particular, the gravitino mass and moduli (we will explain what they are in a moment) are constrained. Finally in a SUSY GUT, baryogenesis is typically accomplished via a process known as leptogenesis. SUSY GUTs, when spontaneously broken to the SM gauge group, typically create magnetic monopoles. There are serious constraints on the present magnetic monopole density. Most of these issues will be discussed in this chapter.

An inflationary early universe was proposed to solve the cosmological flatness and homogeneity problems. It was also offered as a solution to the magnetic monopole problem. Monopoles are generically produced via the Kibble mechanism when one spontaneously breaks $SU(5)$ or $SO(10)$ to the SM gauge group. Assume however one goes through the GUT phase transition prior to inflation, then the monopole density which was produced is diluted during inflation. Then, as long as the reheat temperature is less than the GUT scale, the monopoles are never re-created. Recent results on the polarization of the microwave background suggest that the energy density during inflation was of order a GUT scale. In this case we may have two, not so independent, sectors of the theory, i.e. the Higgs sector responsible for GUT symmetry breaking and the inflaton sector responsible for inflation. Assuming that the energy scale during inflation is of order the GUT scale, then as the universe cools both effects will happen simultaneously. The universe will inflate and at the same time go through the GUT symmetry breaking. Thus any monopoles which may be produced will be diluted by inflation. Clearly the monopole problem is naturally solved in this case.

After inflation the universe reheats and produces a thermal population of particles with mass less than the reheat temperature and interaction rates which are large enough to quickly push these particles into a state of chemical and kinetic

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equilibrium. This last caveat applies particularly to particles with weak interactions whose rate is suppressed by powers of m_Z^{-1}. Gravitinos, on the other hand, interact with rates suppressed by m_{Pl}^{-1}. This means that upon reheating the abundance of gravitinos is sensitive to the precise reheat temperature. Moduli are scalar fields defined by flat directions in the potential. Such fields typically are consequences in many string models, but they were first introduced in supergravity models. Consider the Polonyi field, X, with scalar potential determined by the superpotential, Eq. (7.13), and Kähler potential, Eq. (7.14). The Polonyi field couples to matter only through gravitationally suppressed interactions, which is a common feature of moduli. Again, assuming inflation, the abundance of moduli depends sensitively on the reheat temperature. If the reheat temperature is large enough they can both appear with a thermal abundance.

We now argue that a thermal abundance of either gravitinos or moduli can cause cosmological problems, unless their mass is suitably constrained.

Brief Review of Standard "Big Bang" Cosmology [244, 245]

Using units such that $\hbar = c = k_B = 1$ we define the Planck scale, $M_{Pl} \equiv G_N^{-1/2} = 1.9 \times 10^{19}$ GeV, and the reduced Planck scale $m_{Pl} \equiv \frac{M_{Pl}}{\sqrt{8\pi}} = 2.4 \times 10^{18}$ GeV.

We assume space-time is described, on large scales, by the Friedmann–Robertson–Walker metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{11.1}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$, and the parameter, k, determines the spatial topology of the universe: $k = +1$, closed; $k = 0$, open and flat; $k = -1$, open and curved. $a(t)$ is the cosmological scale parameter. Recall that the FRW metric is uniquely determined by requiring space to be both homogeneous and isotropic. Note that this is the way it appears today on large scales and it can be explained by an early inflationary epoch.

The dynamical equations of cosmology are given by:

Einstein's equation

$$G_{\mu\nu} = \frac{1}{m_{Pl}^2} T_{\mu\nu}, \quad T^{\mu\nu}{}_{;\nu} = 0, \tag{11.2}$$

with $\dot{a} \equiv \frac{da}{dt}$, $\dot{\rho} \equiv \frac{d\rho}{dt}$ implies

$$\begin{aligned} H^2(t) \equiv \left(\tfrac{\dot{a}}{a}\right)^2 &= \tfrac{\rho}{3m_{Pl}^2} - \tfrac{k}{a^2}, \\ \frac{\ddot{a}}{a} &\equiv -(\rho + 3p)/(6m_{Pl}^2), \\ \dot{\rho} &\equiv \quad -3(\rho + p)\, H, \end{aligned} \tag{11.3}$$

where

$$T_{\mu\nu} \equiv (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{11.4}$$

$$u^\mu \equiv \frac{dx^\mu}{d\tau} = (1, \vec{0}). \tag{11.5}$$

ρ, p is the energy, pressure densities, respectively, for a perfect fluid, u^μ is the fluid velocity field 4-vector, and $H(t)$ is the Hubble expansion rate. Note, ρ and p is the total energy and pressure which is a sum of many components. The geodesic equations of motion for a particle (or fluid element) is given by

$$\frac{du^\gamma}{d\tau} + \Gamma^\gamma_{\mu\nu} u^\mu u^\nu = 0. \tag{11.6}$$

In fact, the FRW coordinate position, r, θ, ϕ in Eq. (11.1) describes a coordinate frame which is co-moving with the fluid elements. The distance between two such elements is then increasing with the scale factor $a(t)$.

The dynamical equations, Eq. (11.3), must be supplemented with an equation of state, $p = p(\rho)$. We consider three forms:

1. radiation;

$$p_r = \frac{1}{3}\,\rho_r, \tag{11.7}$$

   [using (11.3) implies]

$$\rho_r \sim \frac{1}{a^4}.$$

2. non-interacting conserved massive particles, i.e. matter;

$$p_m = 0, \tag{11.8}$$

   [using (11.3) implies]

$$\rho_m \sim \frac{1}{a^3}.$$

3. cosmological constant;

$$p_\Lambda = -\rho_\Lambda. \tag{11.9}$$

[using (11.3) implies]

$$\rho_\Lambda = \text{constant}.$$

The cosmological constant Λ is typically defined by

$$\Lambda \equiv \frac{\rho_\Lambda}{m_{Pl}^2}. \tag{11.10}$$

We now have a complete system of equations, Eqs. (11.3), (11.7)–(11.9).

Let us now discuss some observed properties of the present universe which shall be useful later.

1. The present value of the Hubble expansion parameter using Planck data, Table 4 in [246] at 68% CL (TT + low P + lensing)

$$H_0 = h\, 100\,\text{km/s/Mpc}, \tag{11.11}$$

with $h = 0.6781 \pm 0.0092$ (1 pc = 3.26 light years, 1 Mpc $\approx 3 \times 10^{24}$ cm). H_0 determines the observed red shift, $z = \frac{\lambda_{\text{observed}} - \lambda_{\text{emitted}}}{\lambda_{\text{emitted}}} \left(= \frac{a(t_0)}{a(t_e)} - 1\right)$, of distant stars using the relation

$$d = H_0^{-1}\left[z + \frac{1}{2}(1 - q_0)z^2 + O(z^3)\right], \tag{11.12}$$

where d is the present distance to the star and q_0 is the present value of the deceleration parameter, $q_0 = -\left[\frac{\ddot{a}\,a}{\dot{a}^2}|_{\text{today}}\right] \simeq (1 + 3p/\rho)/2$.

2. We define the critical energy density

$$\rho_c(t) \equiv 3\, H^2(t)\, m_{Pl}^2. \tag{11.13}$$

Using Eq. (11.3) we see that for

$$\rho \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \rho_c, \quad k = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}. \tag{11.14}$$

Hence we find today

$$\rho_c^0 \simeq 2 \times 10^{-29}\,\mathrm{g/cm^3}h^2 \simeq (3 \times 10^{-3}\,\mathrm{eV})^4 h^2. \tag{11.15}$$

3. The observed energy density in 2.7 K black body radiation is

$$\rho_r^0 = 4.5 \times 10^{-34}\,\mathrm{g/cm^3}. \tag{11.16}$$

Cosmologists typically define the ratio, for matter of type i,

$$\Omega_i \equiv \frac{\rho_i}{\rho_c}. \tag{11.17}$$

Thus

$$\Omega_r^0 = 2.25 \times 10^{-5} h^{-2}. \tag{11.18}$$

For luminous matter (baryons) we have [Table 4 in [246] at 68% CL (TT + low P + lensing)]

$$\Omega_b^0 h^2 = 0.02226 \pm 0.00023. \tag{11.19}$$

However, the dominant form of energy in the universe is not luminous [again see Table 4 in [246] at 68% CL (TT + low P + lensing)]. It is divided into a dark matter component with

$$\Omega_m^0 = 0.308 \pm 0.012, \tag{11.20}$$

and a dark energy component with

$$\Omega_\Lambda^0 = 0.692 \pm 0.012. \tag{11.21}$$

In addition, the universe appears to be flat with $\Omega_k \simeq 0$. Hence, $\sum_i \Omega_i = 1$.

Note, $\Omega_\Lambda < 1$ implies the observed cosmological constant Λ satisfies $\Lambda < 10^{-120} m_{Pl}^2$. This is the cosmological constant problem.

4. The ratio of the number of baryons to photons is given by

$$\frac{n_b}{n_\gamma}|_0 \simeq 10^{-9} \tag{11.22}$$

where $n_\gamma^0 \simeq 400\,\mathrm{cm^{-3}}$ and $n_b^0 \simeq \rho_b^0/1\,\mathrm{GeV}$.

Relevant Statistical Mechanics

For a particle in thermal equilibrium, the phase space density $f(\overrightarrow{p})$ is given by

$$f(\overrightarrow{p}) \equiv \frac{dN}{d^3\overrightarrow{p}\, d^3\overrightarrow{x}} = \frac{1}{(2\pi)^3} \frac{1}{exp(\frac{E-\mu}{T}) + \theta} \tag{11.23}$$

where

$$\theta = \begin{Bmatrix} +1 & \text{fermions} \\ -1 & \text{bosons} \end{Bmatrix}, \tag{11.24}$$

$E = \sqrt{\overrightarrow{p}^2 + m^2}$, T is the temperature and μ is the chemical potential.

The number density n and energy density ρ are then given by

$$\begin{aligned} n = &\quad N_h \int d^3\overrightarrow{p}\, f(\overrightarrow{p}) \\ \rho = &\ N_h \int d^3\overrightarrow{p}\ E(\overrightarrow{p}) f(\overrightarrow{p}) \equiv n\langle E\rangle, \end{aligned} \tag{11.25}$$

where $\langle E\rangle$ denotes the thermal average and N_h is the number of helicity states. For $T \gg m$, we obtain the number and energy density appropriate for "radiation"

$$\begin{aligned} n_r = &\ \tfrac{g'\ \zeta(3)\ T^3}{\pi^2}, \\ \rho_r = &\ g\ \tfrac{\pi^2}{30}\ T^4 \end{aligned} \tag{11.26}$$

where $g' = N_{hB} + \frac{3}{4}\ N_{hF}$, $g = N_{hB} + \frac{7}{8}\ N_{hF}$ are weighted sums of boson (N_{hB}) and fermion (N_{hF}) helicities, and $\zeta(3) = \sum_{i=1}^{3}\ i^{-3} \sim 1.2$. Note, that the thermal averaged photon energy is

$$\langle E_\gamma\rangle \simeq 2.6\ T, \tag{11.27}$$

and the entropy density is given by

$$S_r = g\frac{2\pi^2}{45}\ T^3. \tag{11.28}$$

For $T \ll m$ we have the number and energy densities for non-relativistic "matter" with $\mu = 0$

$$\begin{aligned} n_m = &\ N_h\ (\tfrac{mT}{2\pi})^{3/2}\ e^{-m/T} \\ \rho_m = &\ m\, n_m + \tfrac{3}{2}\, n_m\, T,\ \ p_m = n_m\, T, \end{aligned} \tag{11.29}$$

where p_m is the pressure density.

Comparing Eqs. (11.7) and (11.26) we see that

$$T \sim \frac{1}{a} \tag{11.30}$$

for radiation in thermal equilibrium. For pressureless matter, out of chemical equilibrium, we have $n_m \propto 1/a^3$. However, for matter in thermal equilibrium, n_m is Boltzmann suppressed ($\propto e^{-m/T}$) and thus decreases much faster than $1/a^3$. It is not always possible though for the reaction rates, which govern the approach to equilibrium, to keep up with the expansion and thus as result the system goes out of chemical equilibrium. Let us now briefly discuss the approach to thermal equilibrium.

Consider two species of particles, x (which is out of thermal equilibrium), and r (in thermal equilibrium). We want to know how x approaches equilibrium [247]. The relativistic version of Boltzmann's equation is given by

$$\begin{aligned} \frac{dn_x}{dt} + 3\frac{\dot{a}}{a}\, n_x = {} & \Lambda^x_{r_1 r_2}[f_{r_1}\, f_{r_2}\, (1-\theta_x f_x)\, |\mathscr{M}(r_1\, r_2 \to x)|^2 \\ & -f_x\, (1-\theta_{r_1} f_{r_1})\, (1-\theta_{r_2} f_{r_2})\, |\mathscr{M}(x \to r_1\, r_2)|^2] \\ & +2\Lambda^{34}_{12}\, [f_{r_1}\, f_{r_2}\, (1-\theta_x f_{x_3})\, (1-\theta_x f_{x_4})\, |\mathscr{M}(r_1\, r_2 \to x_3\, x_4)|^2 \\ & -f_{x_1}\, f_{x_2}\, (1-\theta_{r_3} f_{r_3})\, (1-\theta_{r_4} f_{r_4})\, |\mathscr{M}(x_1\, x_2 \to r_3\, r_4)|^2], \end{aligned} \tag{11.31}$$

where the Lorentz invariant phase space factor Λ is defined by

$$\begin{aligned} \Lambda^{b_1 b_2}_{a_1 a_2} = {} & N_{ha_1}\frac{d^3 p_{a_1}}{(2\pi)^3 2E_{a_1}}\, N_{ha_2}\frac{d^3 p_{a_2}}{(2\pi)^3 2E_{a_2}}\, N_{hb_1}\frac{d^3 p_{b_1}}{(2\pi)^3 2E_{b_1}}\, N_{hb_2}\frac{d^3 p_{b_2}}{(2\pi)^3 2E_{b_2}}\, (2\pi)^4 \\ & \delta^4(\sum_i p_{a_i} - \sum_i p_{b_i}), \end{aligned} \tag{11.32}$$

$\theta_{r,x}$ is defined in Eq. (11.24) and $\mathscr{M}$ is the scattering amplitude for the indicated process. The four terms in Eq. (11.31) represent the four processes in Fig. 11.1, respectively. If we assume that the interactions are both CP and CPT invariant, use the fact that f_r is an equilibrium distribution and assume that the gas is non-degenerate, we obtain

$$\begin{aligned} \frac{dn_x}{dt} + 3\frac{\dot{a}}{a}\, n_x = {} & (n_x^{eq} - n_x)\, \langle \Gamma_{x \to r_1\, r_2} \rangle \\ & +2\, [(n_x^{eq})^2 - (n_x)^2]\, \langle \sigma_{xx \to rr}\, v \rangle. \end{aligned} \tag{11.33}$$

n_x^{eq} is the thermal equilibrium number density. Note, that equilibrium is achieved IFF $H = \frac{\dot{a}}{a} \ll \langle \Gamma \rangle$ or $\langle \sigma\, v \rangle$, i.e. the expansion rate is less than a typical reaction rate. Otherwise we find $n_x \sim 1/a^3$, as in Eq. (11.8).

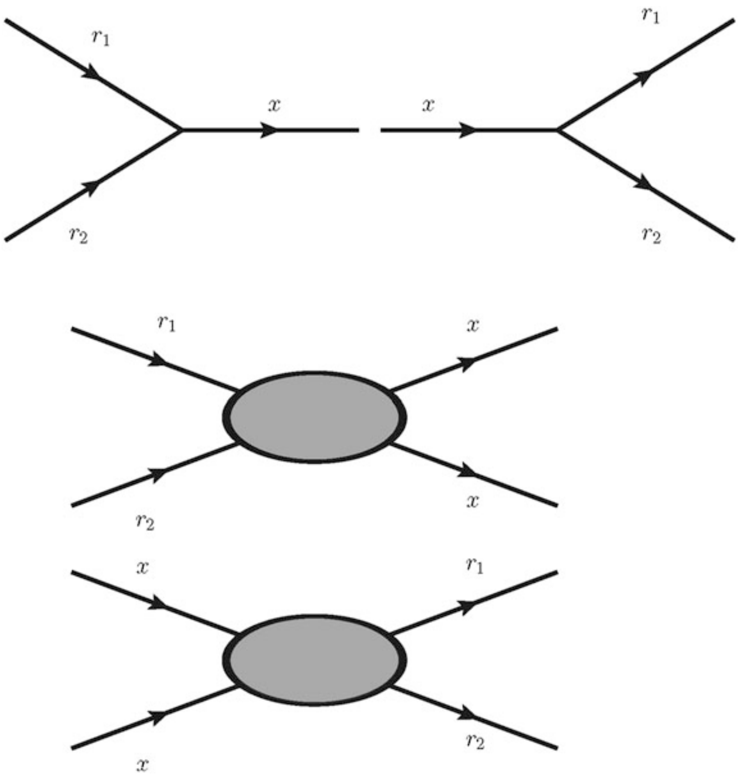

Fig. 11.1 Scattering processes affecting thermal equilibration

We now have the tools to review the standard picture of the early universe. In Table 11.1 we have listed the major events. We assume that all particles are in thermal equilibrium at the Planck temperature. As the universe expands it goes through several phase transitions. There is assumed to be an inflationary phase. The latest results from the Bicep2-Keck-Planck re-analysis [248] of CMB polarization data suggest that the energy density during the inflationary epoch was of order the GUT scale. Typically during this transition the universe cools adiabatically and then reheats to a temperature, T_{reheat}. At this point we reset our clocks and start the thermal history anew. We won't discuss the inflationary model.[1] Suffice it to say that it must solve the cosmological problems of homogeneity, isotropy, flatness and the monopole problem of GUTs (for monopoles in GUTs, see for example [251]).

[1]A possible model of inflation with the gauge symmetry breaking scale determining the energy density during inflation is given in [249, 250].

Table 11.1 Thermal history of the Universe

Time (s)	T (GeV)	Comments
10^{-44}	10^{19}	"Initial conditions" assume all particles in thermal equilibrium
10^{-38}	10^{16}	GUT scale/inflation
	Reheat $\Rightarrow$ reset clocks	
10^{-28}	10^{11}	Fundamental SUSY breaking scale (gravity mediation)
10^{-18}	10^{6}	Fundamental SUSY breaking scale (low energy gauge mediation)
10^{-10}	10^{2}	Electroweak breaking scale ($g \geq 100$)
10^{-5}	$1/3$	QCD phase transition from quark-gluon plasma to hadrons
10^{-2}	10^{-2}	ν_μ decouple : $\langle \sigma v n_{\nu_\mu} \rangle < H$
		$\frac{n_n}{n_n+n_p}$: $1/2 \rightarrow 1/6$: $e^{-(m_n-m_p)/T}$
	2×10^{-3}	ν_e decouple : $\langle \sigma v n \rangle < H$
4	$1/2 \times 10^{-3}$	$e^+ e^-$ annihilate : $e^{-m_e/T}$
		$(4(7/8) + 2)T_\nu^3 = 2\, T_\gamma^3$ (entropy conservation) $\Rightarrow T_\nu = (4/11)^{1/3}\, T_\gamma$
100	10^{-4}	"Big Bang" Nucleosynthesis
$10^{12} = 3 \times 10^4$ years	$10^{-9}\ \Omega_m^0/\Omega_r^0\ T_0 \sim 3.18 \times 10^{-9}\ h^2$	$T = T_{eq}$: universe becomes matter dominated
	$1/3 \times 10^{-9}$	T = T$_{recombination}$: $p + e \rightarrow$ neutral hydrogen and photons decouple

By the time the universe gets down to a temperature of order 100 MeV, particles with mass greater than this temperature are severely Boltzmann suppressed or are out of equilibrium. At about 100 MeV, muons begin to annihilate [252]. Neutrinos will soon decouple, leaving an asymptotic ratio of neutrons to neutrons plus protons of about 1/6. This is the initial ratio for the subsequent process of helium synthesis (at the temperature of $T_{H_e} \sim 0.1$ MeV) where all the remaining neutrons are incorporated into helium nuclei. During this epoch, also known as nucleosynthesis, we have the following processes occurring, $(p + n \leftrightarrow d + \gamma;\ \ d + d \leftrightarrow H_e^3 + n \leftrightarrow H^3 + p;\ \ H^3 + d \leftrightarrow H_e^4 + n)$ where d denotes deuterium. At the end, all free neutrons are depleted into H_e^4. Thus at the end of this epoch the thermal bath includes, $\{p,\ H_e^4,\ d,\ \gamma,\ \nu,\ e\}$ with a ratio of $d/H_e^4 \sim 10^{-5}$. Then at a temperature $T \sim 1/2$ MeV, electrons annihilate and heat up the photons but not the neutrinos, since they have already decoupled. Using entropy conservation we find that the neutrinos are colder than the photons with $T_\nu = (\frac{4}{11})^{1/3}\, T_\gamma$.

Under the standard thermal history of the universe, throughout the epoch from T_{Pl} to $T_{eq} \sim (\Omega_m^0/\Omega_r^0)\ T_0 \sim 3.18\ h^2\,\text{eV} \sim 1.46\,\text{eV}$ [i.e. at time t_{eq} when $\rho_r(t_{eq}) = \rho_m(t_{eq})$], the universe is radiation dominated (i.e. the first 3×10^4 years). Thereafter the energy density is matter dominated. At about $T \sim 1/3\,\text{eV}$, free protons and electrons combine to form neutral hydrogen. This is the so-called recombination temperature. Once this occurs, photons decouple. These photons continue to red-shift for the next $\sim 10^{10}$ years. They are presently observed as the Cosmic Microwave Background radiation with $T_0 \sim 2.7\,\text{K}$.

11.1 Baryogenesis via Leptogenesis

The observed universe has more baryonic matter than anti-baryonic matter. In SUSY GUTs, baryon number minus lepton number, $B-L$, is conserved, but separately both B and L are violated at the GUT scale. If we assume that, at the Planck temperature, we have $n_B = n_L = 0$, i.e. an equal number of baryons and anti-baryons and leptons and anti-leptons, then we have for all time $n_B - n_L = 0$. What ingredients are necessary in order to generate a non-vanishing net baryon asymmetry? This was discussed many years ago by Sakharov [253]. One needs to violate baryon number and both C and CP and to do it while out of thermal equilibrium. Because in thermal equilibrium CPT guarantees that for every baryon there is an anti-baryon and baryon number and lepton number violation will always drive the system to thermal equilibrium with $n_B = n_L = 0$.

It was first suggested that the X and Y gauge bosons, when they decay out of the thermal bath, could generate the non-vanishing baryon number, since their interactions violate both B and L separately (yet conserving $B - L$), and non-vanishing phases in the CKM matrix elements violate CP. They go out of equilibrium when $T \leq M_X$, assuming $\Gamma_{(X,Y)} < H$. When $T > M_X$ their number density is of order T^3. However, when $T < M_X$ their equilibrium number density is Boltzmann suppressed and the decay rate cannot keep up with the equilibrium number density which is going to zero exponentially as $(e^{-M_x/T})$. Hence when they decay they quickly go out of chemical equilibrium.

Note however, that the electroweak theory also violates both B and L, preserving $B-L$ via sphaleron interactions [254, 255]. These $\Gamma_{(B+L)}$ interactions occur at a rate which typically satisfies $\Gamma_{(B+L)} > H$. They thus drive any non-vanishing $n_B = n_L$ (due to $X,\ Y$ decay) to zero [256]. Thus in order to generate a non-vanishing baryon number one needs to violate $B - L$ and, hence, generate a non-vanishing $n_B \neq n_L$ [257, 258]. This is possible in a SUSY GUT with large Majorana masses for sterile neutrinos. When the heavy sterile neutrinos, responsible for the See-Saw mechanism which generates three light active neutrinos, decay out of thermal equilibrium, they can violate C, CP and L (since they are Majorana neutrinos). The necessary conditions to obtain the observed ratio n_b/n_γ was given in a nice review article [259]. Their analysis assumes thermal leptogenesis and requires the lightest Majorana neutrino with mass, M_{N_1} of order $10^{10}\,\text{GeV}$ and a reheat temperature

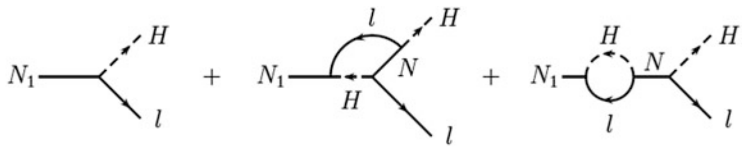

Fig. 11.2 Decay amplitudes for the lightest Majorana neutrino, N_1, which lead to non-zero ϵ due to phases in the Yukawa matrices

above M_{N_1}. *CP* violation occurs when the Majorana neutrino decays and the lepton number asymmetry produced per decay is given in terms of the Dirac Yukawa matrices coupling the heavy right-handed sterile neutrino to the lepton and Higgs weak doublets. These Yukawa matrices typically have non-vanishing phases which lead to a *CP* asymmetry of the form (see Fig. 11.2)

$$\epsilon_1 = \frac{\Gamma_{N_1 \to l\ H_u} - \Gamma_{N_1 \to l^*\ H_u^*}}{\Gamma_{N_1 \to l\ H_u} + \Gamma_{N_1 \to l^*\ H_u^*}} \tag{11.34}$$

They find

$$\epsilon_1 \simeq \frac{3}{16\pi} \frac{1}{(Y_\nu\ Y_\nu^\dagger)_{11}} \sum_{i=2,3} Im[(Y_\nu\ Y_\nu^\dagger)_{1i}^2]\ \frac{M_{N_1}}{M_{N_i}}. \tag{11.35}$$

This is the simplest scenario. It neglects the heavier Majorana neutrinos which might also generate a lepton number asymmetry, depending on the value of the reheat temperature. It also does not describe the possible non-thermal leptogenesis. But this is more model dependent. Unfortunately, without a model, there is no direct connection between the sign of the *CP* violation in the heavy Majorana sector and *CP* violation which might be observable for the light active neutrinos. On the positive side, however, given a SUSY GUT which fits low energy fermion masses and mixing angles (including neutrinos), it also has the possibility for predicting the observed baryon to photon ratio.

11.2 Bounds on Masses of Gravitationally Coupled Particles

11.2.1 Mass Bounds for a Heavy Gravitino: No Inflation

Let us first assume that gravitinos are the heaviest supersymmetric particle in a SUSY GUT. The scattering rate for gravitinos with the thermal bath at temperature T is generically given by

$$\langle \sigma\ v\ n \rangle \sim \alpha N T^3 / m_{Pl}^2 \tag{11.36}$$

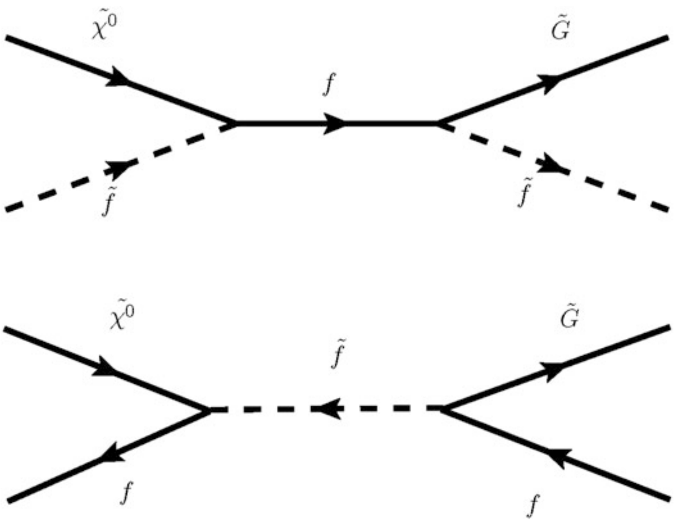

Fig. 11.3 Gravitinos can come back into thermal equilibrium scattering off other SUSY (or non-SUSY) particles in the thermal bath. The number density of gravitinos after reheating depends on the reheat temperature. These two graphs just represent two of all the possible permutations

(see Fig. 11.3) where $\alpha \sim 10^{-2}$ is of order the electromagnetic coupling strength, and N is the number of particles in the thermal bath. Gravitinos decouple from the thermal bath when the interaction rate becomes less than of order the expansion rate of the universe, i.e. when

$$\alpha N T^3_{decoupling}/m^2_{Pl} \sim \langle \sigma \; v \; n \rangle \lesssim H \sim (g\pi^2/90)^{1/2} T^2_{decoupling}/m_{Pl} \tag{11.37}$$

or with $g \sim N \sim 10^2$ we have

$$T_{decoupling} \lesssim m_{Pl}. \tag{11.38}$$

Thus any period of inflation below the Planck temperature will dilute the gravitino number density so much so that you might expect that there are almost no gravitinos left today. However, that is not quite correct, since once the universe reheats after inflation, gravitinos can be produced by the same thermal scattering processes.

In order to better understand what is going on and to set up the problem, we first discuss bounds on a heavy gravitino, neglecting inflation [239]. The dimension-five operator coupling the helicity $\pm 3/2$ components of the gravitino[2] to a vector boson

[2]The gravitino is described by a Rarita-Schwinger spin 3/2 field, $\tilde{G}_\mu$.

A_μ and its gaugino partner λ is

$$(1/4m_{Pl})\bar{\lambda}^a\gamma^\mu\sigma^{\nu\rho}\tilde{G}_\mu F^a_{\nu\rho} + h.c., \tag{11.39}$$

and the dimension-five operator coupling the helicity $\pm 3/2$ components of the gravitino to a chiral scalar ϕ_i and its fermionic partner ψ_{iL} is

$$(1/2m_{Pl})\overline{\tilde{G}_\mu}\gamma^\nu\partial_\nu\phi^{*i}\gamma^\mu\psi_{iL} + h.c.. \tag{11.40}$$

Since the gravitino coupling is Planck suppressed, the lifetime is long. The gravitino can decay into all SUSY particles consistent with energy conservation. The gravitino decay rate is of order[3]

$$\Gamma_{\tilde{G}} \sim \frac{N}{2\pi}\frac{m_{3/2}^3}{m_{Pl}^2} \tag{11.41}$$

where N is the number of SUSY states lighter than $m_{3/2}$. And its lifetime is of order

$$\tau_{\tilde{G}} = 1/\Gamma_{\tilde{G}} \sim 4\times 10^5\,(\frac{100\,\text{GeV}}{m_{3/2}})^3\text{s}. \tag{11.42}$$

Assuming gravitinos were in thermal equilibrium when they decoupled, their number density is given by the radiation number density $n_{\tilde{G}} = \frac{3\,\zeta(3)\,T^3}{2\pi^2}$ with $T \sim m_{Pl}$. Therefore the ratio of the number of gravitinos to photons is $Y_{\tilde{G}} = 3/4$. This ratio is preserved to low temperature.[4] Finally gravitinos decay out of the system when $H \lesssim \Gamma_{\tilde{G}}$. Note, however, that at temperatures below the gravitino mass, the universe becomes matter dominated and remains matter dominated until gravitinos decay out of the system. Therefore we have

$$\Gamma_{\tilde{G}} = \frac{N}{2\pi}\frac{m_{3/2}^3}{m_{Pl}^2} = \sqrt{\frac{m_{3/2}\,\zeta(3)\,T_{decay}^3}{2\pi^2 m_{Pl}^2}} \tag{11.43}$$

or

$$T_{decay} \simeq \left(N\sqrt{\frac{1}{2\,\zeta(3)}}\frac{m_{3/2}^{5/2}}{m_{Pl}}\right)^{2/3} \sim 20\left(\frac{m_{3/2}}{100\,\text{GeV}}\right)^{5/3}\ \text{eV}. \tag{11.44}$$

[3]This is given simply by dimensional analysis.

[4]This is not quite true since every time some particles annihilate out of the thermal bath, photons get heated up while the gravitinos don't.

When gravitinos decay out of the system all their energy is converted into radiation. The universe then reheats to a temperature, T_{reheat} given by energy conservation, i.e.

$$\rho_{\tilde{G}}(T_{decay}) = \frac{m_{3/2}\ 3\ \zeta(3)\ T_{decay}^3}{2\pi^2} = \frac{g\pi^2 T_{reheat}^4}{30} = \rho_{rad}(T_{reheat}) \tag{11.45}$$

or

$$\Gamma_{\tilde{G}} = \frac{N}{2\pi}\frac{m_{3/2}^3}{m_{Pl}^2} = \sqrt{\frac{g\pi^2 T_{reheat}^4}{90\ m_{Pl}^2}}. \tag{11.46}$$

We obtain

$$T_{reheat} = \sqrt{\Gamma_{\tilde{G}}\ m_{Pl}(\frac{90}{g\pi^2})^{1/2}} \sim 1.4\left(\frac{m_{3/2}}{100\,\text{GeV}}\right)^{3/2}\ \text{keV}. \tag{11.47}$$

This is below the temperature of order 1–10 MeV when nucleosynthesis begins with the "correct" neutron to proton ratio determined by the expansion rate. But now the universe expands too quickly and the analysis of nucleosynthesis fails. This is a major problem. The solution is that gravitinos have mass

$$m_{3/2} \geq 30\ \text{TeV}, \tag{11.48}$$

implying a reheat temperature of order 7 MeV. Of course, the actual bound depends on the details. Note, however, with $m_{3/2} \geq 30$ TeV there is a huge amount of entropy released when gravitinos decay. We have $\Delta = (\frac{T_{reheat}}{T_{decay}})^3 \sim 2\times 10^4$. This will severely dilute the baryon number density.

11.2.2 Mass Bounds for a Heavy Gravitino: Assuming Inflation

Now we can address the issue of inflation. Assume that at about a temperature of order the GUT scale, the universe goes through a period of inflation. Then the primordial gravitinos will dilute away. After inflation the universe reheats. Then we must calculate the gravitino number density produced at the reheat temperature. If $Y_{\tilde{G}} = \frac{n_{\tilde{G}}}{n_\gamma}$ is small enough, such that when gravitinos decay they no longer dominate the energy density of the universe, there won't be a problem with nucleosynthesis. This occurs when $m_{3/2}Y_{\tilde{G}} \lesssim 10\,\text{MeV}$ or $Y_{\tilde{G}} \lesssim 10^{-4}\left(\frac{100\,\text{GeV}}{m_{3/2}}\right)$.

Unfortunately, this is not sufficient to solve all the problems with late decaying gravitinos. When they decay (after nucleosynthesis), their decay products have lots of energy. In particular, energetic hadrons and photons can disrupt the deuterium produced during nucleosynthesis, since the deuterium binding energy is quite small, of order 2.3 MeV. This gives a strong bound on the reheat temperature [260]. However the most stringent bound comes from the photo-disintegration of ^{4}He via processes such as $\gamma +^4 He \rightarrow n +^3 He$ and $\gamma +^4 He \rightarrow p +^3 H$ (with $^3H \rightarrow^3 He + e + \bar{\nu}_e$) which then overproduces deuterium and ^{3}He. This bound requires [261]

$$Y_{\tilde{G}} < 6.1 \times 10^{-14} \, (m_{3/2}/100\,\text{GeV})^{-1} \tag{11.49}$$

and a reheat temperature,

$$T_{MAX} \equiv T_{reheat} < 2.5 \times 10^8 \, (m_{3/2}/100\,\text{GeV})^{-1}\,\text{GeV}. \tag{11.50}$$

This result is obtained in [260] by using the Boltzmann equation

$$\frac{dn_{\tilde{G}}}{dt} \simeq \langle \Sigma_{tot} v \rangle \, (\frac{2\,\zeta(3)}{\pi^2} T^3)^2 \tag{11.51}$$

with $\langle \Sigma_{tot} v \rangle = \sum_{ij} \sigma_{ij}^{tot} \, v \, \frac{g_i' \, g_j'}{4} = m_{Pl}^{-2}(15.59\, g_3^2 + 5.25\, g_2^2 + 1.65\, g'^2)$ where $\{i, j\}$ are all possible initial states and $g_i' = (N_{hB})_i + \frac{3}{4}(N_{hF})_i$. The total number g of degrees of freedom in the MSSM is $g = \sum_B N_{hB} + \frac{7}{8} \sum_F N_{hF} = 122(1 + \frac{7}{8}) = \frac{915}{4}$.[5] Using $t = \sqrt{\frac{90\, m_{Pl}^2}{4\,\pi^2\, g}}/T^2$, we have

$$\frac{dn_{\tilde{G}}}{dT} \simeq -2\langle \Sigma_{tot} v \rangle \, (2\zeta(3)/\pi^2)^2 \, (\frac{90}{4\,\pi^2\, g})^{1/2} \, m_{Pl} \, T^3 \tag{11.52}$$

which gives

$$n_{\tilde{G}}(T_{MAX}) \sim \langle \Sigma_{tot} v \rangle \, (2\zeta(3)/\pi^2)^2 \, (\frac{90}{16\,\pi^2\, g})^{1/2} \, m_{Pl} \, T_{MAX}^4 \tag{11.53}$$

where the integral is dominated by temperatures of order T_{MAX}. As temperatures decrease the gravitino number density then decreases and at low temperatures we have

$$n_{\tilde{G}}(T_{\tilde{G}}) \sim \langle \Sigma_{tot} v \rangle \, (2\zeta(3)/\pi^2)^2 \, (\frac{90}{16\,\pi^2\, g})^{1/2} \, m_{Pl} \, T_{MAX} T_{\tilde{G}}^3. \tag{11.54}$$

Plugging in the values for $\alpha_i(T_{MAX})$ given in [260], they obtain

$$n_{\tilde{G}}(T_{\tilde{G}}) \sim 3.35 \times 10^{-12} \, (T_{MAX}/10^9\,\text{GeV}) \, (1 - 0.018 \ln(T_{MAX}/10^9\,\text{GeV})) \, T_{\tilde{G}}^3. \tag{11.55}$$

[5] This result neglects the right-handed neutrinos needed for the See-Saw mechanism. But assuming they are heavy, they can safely be ignored.

This gives

$$Y_{\tilde{G}} \sim (2\zeta(3)/\pi^2)^{-1}\, 3.35\times10^{-12}\, (T_{MAX}/10^9\,\text{GeV})\, (1-0.018\ln(T_{MAX}/10^9\,\text{GeV}))(\frac{T_{\tilde{G}}}{T_\gamma})^3 \tag{11.56}$$

where $T_{\tilde{G}} = (172/10065)^{1/3} T_\gamma$ takes into account the difference in the photon temperature and the gravitino temperature after many particles have gone out of equilibrium. With $T_{MAX} \sim 2.5 \times 10^6$ GeV, we have $Y_{\tilde{G}} \sim 6 \times 10^{-16}$ where we have assumed that the gravitino is the heaviest superparticle with $m_{3/2} = 10$ TeV.

The most recent analysis on bounds on the gravitino mass coming from cosmology are found in [262], based on the previous work in [263, 264]. This paper treats four different benchmark points in the CMSSM (see Table 11.2). It turns out, however, that the upper bound on the reheat temperature for very heavy gravitinos is more or less independent of the case. In most cases, for a gravitino with mass greater than about 40 TeV, the upper bound on the reheat temperature is about 10^{10} GeV (see Fig. 11.4 for Case 3).

Table 11.2 These are the parameters for the four cases studied in [262]

	Case 1	Case 2	Case 3	Case 4
$m_{1/2}$	300 GeV	600 GeV	300 GeV	1200 GeV
m_0	141 GeV	218 GeV	2397 GeV	800 GeV
A_0	0	0	0	0
$\tan\beta$	30	30	30	45
μ	389 GeV	726 GeV	231 GeV	−1315 GeV
$m_{\tilde{\chi}_1^0}$	117 GeV	244 GeV	116 GeV	509 GeV
$\Omega_{LSP}^{(thermal)} h^2$	0.111	0.110	0.106	0.111

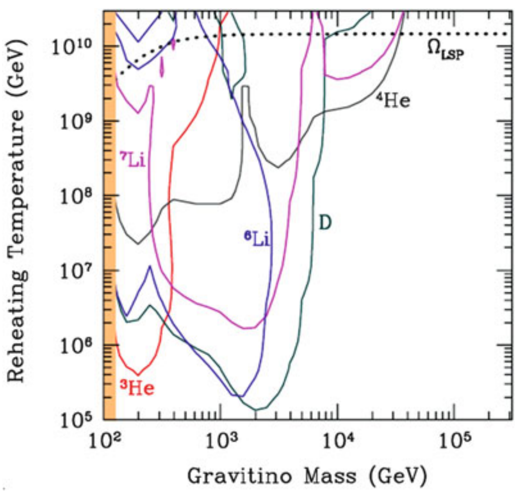

Fig. 11.4 The upper bound on the gravitino mass, $m_{3/2}$, as a function of the reheat temperature, T_R. The figure is taken from [262] for Case 3. The *solid line* denotes the upper bound on the reheat temperature from the closure limit due to the LSP. *Reprinted Fig. 4 with permission from Masahiro Kawasaki, Phys. Rev. D, 78, 065011-6 (2008). Copyright (2008) by the American Physical Society*

11.2.3 Mass Bounds for a Light Gravitino

We now consider the mass bounds on a gravitino LSP where we find a relation in the $m_{3/2}$ vs. T_{reheat} plane [265, 266]. In local supersymmetry the Goldstino becomes the longitudinal component of the gravitino, giving the gravitino a mass

$$m_{3/2} = \frac{F}{\sqrt{3} m_{Pl}} = 2.4 \left(\frac{F}{(100\,\mathrm{TeV})^2} \right) \mathrm{eV}. \tag{11.57}$$

For low energy SUSY breaking, the gravitino is the LSP and the lightest standard model superparticle is then the NLSP, and can decay into its partner and the gravitino. The lowest order coupling of the Goldstino is fixed by the supersymmetric Goldberger-Treiman low energy theorem to be given by [265, 267, 268]

$$\mathscr{L} \supset -\frac{1}{F} j^{\alpha\mu} \partial_\mu \, G_\alpha + h.c. \tag{11.58}$$

where $j^{\alpha\mu}$ is the supercurrent and G_α is the spin 1/2 longitudinal Goldstino component of the gravitino. The decay to the Goldstino component is then suppressed only by F rather than m_{Pl}. In the case that the NLSP is mostly Bino, $\tilde{B}$, the coupling leads to a transition magnetic dipole moment between the NLSP and gravitino [269],

$$\cos\theta_W (m_{\tilde{B}}/2\sqrt{2}F) \tilde{B} \, \bar{\sigma}^\mu \sigma^\nu \; G \, F_{\mu\nu} + h.c. \tag{11.59}$$

producing a decay rate

$$\Gamma(\tilde{B} \to G + \gamma) = \frac{\cos^2\theta_W m_{\tilde{B}}^5}{16\,\pi\, F^2}. \tag{11.60}$$

Since it is not suppressed by m_{Pl} the light Goldstino can remain in equilibrium to much lower temperatures. Using the Goldberger-Treiman type relation for gravitino scattering on matter, Pagels and Primack find a scattering cross-section relation [265]

$$\sigma_G \cdot v \approx \frac{\pi \; M \; E_{threshold}}{F^2} = 10^{-37}\,\mathrm{cm}^2 \; (\frac{E_{threshold}}{100\,\mathrm{GeV}}) \; (\frac{1\,\mathrm{TeV}^2}{F})^2 \tag{11.61}$$

assuming the lightest superparticle mass of order 100 GeV, M is the typical target mass and they take the effective number of states in the thermal bath when Goldstinos decouple, $g_G \sim 100$. The helicity $\pm 3/2$ states of the gravitino interact with gravitational strength and thus they go out of thermal equilibrium at the Planck scale.

Again, in order to set up the problem, let's assume inflation at a GUT scale and discuss the bound on the gravitino mass, neglecting reheating. The spin $\pm 3/2$ components of the gravitino decouple at a temperature of order M_{Pl}, but the spin

$\pm 1/2$ components of the gravitino stay in thermal equilibrium to much lower temperatures. The Goldstino decay constant F determines the gravitino mass and now we show that the gravitino mass has an upper bound such that gravitinos do not overclose the universe today. Recall that presently, photons in the CMB have a temperature $T_0 \simeq 2.7\,\mathrm{K}$ and a number density $n_\gamma^0 = \frac{2\zeta(3)\,{T^0}^3}{\pi^2} \simeq 400/\mathrm{cm}^3$. Neutrinos, on the other hand, are colder with a temperature $T_\nu^0 = (4/11)^{1/3}\,T_\gamma^0$ as a consequence of decoupling prior to electrons and positrons annihilating out of the thermal bath. This heated up the photons without affecting the neutrinos. Similarly, after gravitinos decouple from the thermal bath many more particles annihilate. Thus the effective Goldstino temperature satisfies

$$g_G\,{T_G^0}^3 = 3 \times 2 \times \frac{7}{8}\,{T_\nu^0}^3 + 2\,{T_\gamma^0}^3 = \frac{43}{11}\,{T_\gamma^0}^3 \tag{11.62}$$

and the Goldstino number density is given by

$$n_G^0 = \frac{3}{4}\,\frac{43}{11}\,\frac{1}{g_G}\,n_\gamma^0. \tag{11.63}$$

Requiring that the energy density in Goldstinos is less than or equal the dark matter density, we have

$$m_{3/2}\,n_G^0 \leq \Omega_m\,\rho_c^0 \tag{11.64}$$

implies

$$m_{3/2} \leq \frac{\Omega_m}{\Omega_\gamma}\,\frac{g_G}{\frac{3}{4}\,\frac{43}{11}}\,2.6T^0. \tag{11.65}$$

or

$$m_{3/2} \leq \frac{0.3}{2.25 \times 10^{-5}\,h^{-2}}\,\frac{g_G}{\frac{3}{4}\,\frac{43}{11}}\,6 \times 10^{-4}\,\mathrm{eV} \simeq 1.25\,g_G\,\mathrm{eV}. \tag{11.66}$$

Assuming for the moment $g_G \lesssim 200$ we have

$$m_{3/2} \leq 250\,\mathrm{eV}. \tag{11.67}$$

This also puts an upper bound on the scale of SUSY breaking. We have

$$F = \sqrt{3}\,m_{3/2}\,m_{Pl} \leq (10^3\,\frac{g_G}{200}\,\mathrm{TeV})^2. \tag{11.68}$$

For self-consistency, let's calculate the decoupling temperature for Goldstinos. Using Eq. (11.61), Goldstinos decouple when

$$\sigma_G \cdot v\, n_r = \sigma_G \cdot v(\frac{g'_G \zeta(3) T^3_{decoupling}}{\pi^2}) \lesssim H(T_{decoupling}) = \sqrt{\frac{g_G \frac{\pi^2}{30} T^4_{decoupling}}{3 m^2_{Pl}}} \tag{11.69}$$

or

$$T_{decoupling} \approx (\frac{\sqrt{g_G}\, \pi^3}{g'_G\, \zeta(3)\, \sqrt{90}})\, \frac{1}{\sigma_G \cdot v m_{Pl}} = (\frac{\sqrt{g_G}\, \pi^3}{g'_G\, \zeta(3)\, \sqrt{90}})\, \frac{1}{\sigma_G \cdot v m_{Pl}}. \tag{11.70}$$

Using the bound, Eq. (11.68), we find

$$T_{decoupling} \approx 320\,\text{GeV}. \tag{11.71}$$

At this temperature, the number of particles in thermal equilibrium includes at least all SM particles, giving $g_G = 2\times 45\times 7/8 + 2\times 12 + 4 = 106.75$ degrees of freedom which is more or less consistent with the value we have chosen, i.e. $g_G \sim g'_G = 200$.

Let us now take into account the effects of reheating after inflation. The authors of [266] use the Boltzmann equations

$$\frac{dn_{\tilde{G}}}{dt} + 3Hn_{\tilde{G}} = \langle \Sigma_{tot}\ v\ \rangle\ n^2_{rad} + \sum_i n_i \langle \Gamma_i \rangle \tag{11.72}$$

(taking into account the scattering of gauginos and gauge bosons in the thermal bath) where $\langle \Sigma_{tot}\ v\ \rangle = \frac{1}{24\pi m^2_{3/2}\ m^2_{Pl}}\ (26.0\ g_3^2\ M_3^2 + 9.16\ g_2^2\ M_2^2 + 2.44\ g_1^2\ M_1^2)$ and $\{n_i,\ \ \Gamma_i\}$ are the number density and decay rate of the ith superparticle. Requiring $\rho^0_G\ \leq\ \rho^0_c$ the authors of [266] obtain the following upper bound on the Goldstino mass as a function of the reheat temperature (see Fig. 11.5). The upper bound on the gravitino mass, given by the dotted region in Fig. 11.5, is excluded. The exact value depends on the assumed value of the NLSP mass. The bound obtained is $m_{3/2}\ <\ 3.4$ GeV $(9.3\,\text{GeV},\ 288.5\,\text{GeV},\ 771.5\,\text{GeV})$ for $m_{NLSP} = 50\,\text{GeV}$ $(100\,\text{GeV},\ 500\,\text{GeV},\ 1\,\text{TeV})$.

The most recent paper on the bound on the mass of a gravitino LSP is also given in [262]. The bounds are expressed as a simultaneous constraint on the gravitino mass and NLSP mass. In all cases a light gravitino is allowed. Only in the case of a sneutrino NLSP is a heavier gravitino allowed (see Figs. 11.6, 11.7 and 11.8).

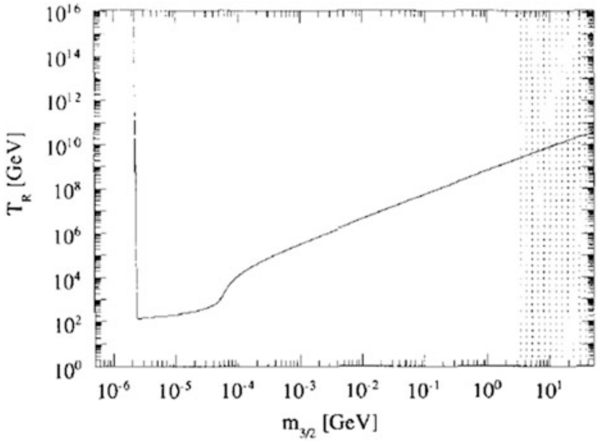

Fig. 11.5 The upper bound on the gravitino mass, $m_{3/2}$, as a function of the reheat temperature, T_R, after inflation; assuming the light gravitino LSP doesn't overclose the universe. The figure is taken from [266]. They took all the squark and slepton masses to be 1 TeV, gaugino masses and the NLSP mass = 50 GeV and GUT relations on the gaugino masses are assumed. The *solid line* denotes the upper bound on the reheating temperature from the closure limit. The *dotted region* is excluded from the arguments of the light element photo-destruction if the NLSP has a relic density as large as $m_{NLSP}\ Y_{NLSP} = 5 \times 10^{-11}$ GeV and is assumed to decay radiatively with a lifetime shorter than 5.3×10^6 s. *Reprinted from Phys. Lett. B303, 289 (1993), T. Moroi, H. Murayama and Masahiro Yamaguchi, "Cosmological constraints on the light stable gravitino," Fig. 1, Page 292, Copyright (1993), with permission from Elsevier*

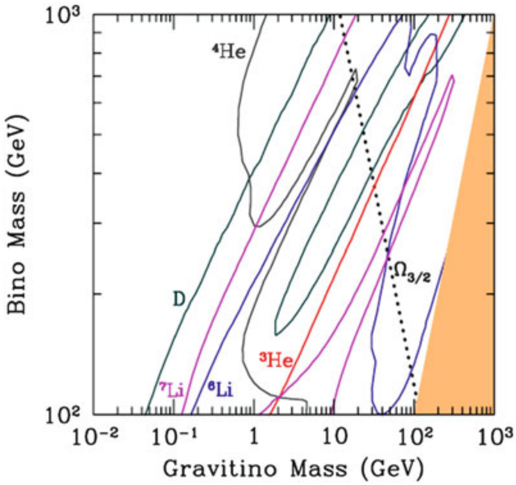

Fig. 11.6 The upper bound on the gravitino mass, $m_{3/2}$, as a function of the lightest bino mass. *Reprinted Fig. 7 with permission from Masahiro Kawasaki, Phys. Rev. D, 78, 065011-8 (2008). Copyright (2008) by the American Physical Society*

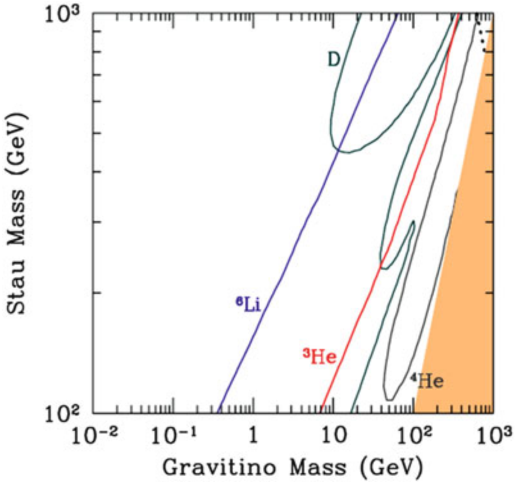

Fig. 11.7 The upper bound on the gravitino mass, $m_{3/2}$, as a function of the lightest stau mass. *Reprinted Fig. 12 with permission from Masahiro Kawasaki, Phys. Rev. D, 78, 065011-10 (2008). Copyright (2008) by the American Physical Society*

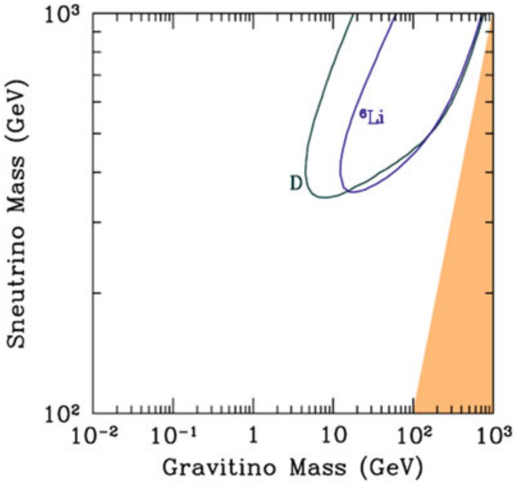

Fig. 11.8 The upper bound on the gravitino mass, $m_{3/2}$, as a function of the lightest sneutrino mass. *Reprinted Fig. 15 with permission from Masahiro Kawasaki, Phys. Rev. D, 78, 065011-12 (2008). Copyright (2008) by the American Physical Society*

11.2.4 Cosmological Moduli Problem

Finally we consider the cosmological moduli problem [240–242, 270, 271]. For a recent review, see [272]. Supergravity and string models tend to have many scalar fields with relatively flat potentials and couplings to visible matter suppressed by the Planck scale. Such scalar fields are called moduli and they cause problems in the early universe. Let's consider the discussion first given in [240]. The simple Polonyi model for supersymmetry breaking in supergravity leads to a cosmological problem. Given the Polonyi superpotential, Eq. (7.13), and Kähler potential, Eq. (7.14), we obtain the zero temperature scalar potential, $V_0(X)$, Eq. (4.11). However in the early universe we must consider the finite temperature free energy

$$V^T(\tilde{X}) = V_0(\tilde{X}) + V_1(\tilde{X}) \quad (11.73)$$

with $\tilde{X} = Re(X)/m_{Pl}$ given by

$$V_0(\tilde{X}) = \mu^4 \ \exp(\tilde{X}^2) \ [1 - 3\ \beta_0^2 + (\beta_0^2 - 1)\ \tilde{X}^2 + 2\beta_0(\tilde{X}^2 - 2)\ \tilde{X} + \tilde{X}^4] \quad (11.74)$$

$$V_1(\tilde{X}) = -\frac{1}{90\pi^2}\ g(T)\ T^4 + \frac{1}{24}\ T^2\ (Tr(M_S^2) + \frac{1}{2}Tr(M_F^2) + 3Tr(M_V^2)) \quad (11.75)$$

$$+[\frac{\Lambda^2}{(4\pi)^2}]\ (Tr(M_S^2) - Tr(M_F^2) + 3Tr(M_V^2)) + O(M^4).$$

where $\tilde{X}_0 = \sqrt{3} - 1$, $\beta_0 = \beta/m_{Pl} = 2 - \sqrt{3}$, and $\mu = \sqrt{m m_{Pl}} \sim 10^{10}$ GeV [see Eq. (7.13)]. Given the values of M_S, M_F (M_V is independent of $\tilde{X}$) the free energy is

$$V^T(\tilde{X}) = V_0(\tilde{X}) + (\kappa + \kappa'(T^2))[V_0(\tilde{X}) + \mu^4\ \exp(\tilde{X}^2)\ (\beta_0 + \tilde{X})^2] \quad (11.76)$$

where $\kappa = [\frac{2(g-1)}{4\pi^2}]\Lambda^2/m_{Pl}^2$ and $\kappa' = \frac{2}{24}(g-1)\ T^2/m_{Pl}^2$.

The minimum of the zero temperature (one loop corrected) potential is shifted to [273]

$$\langle\tilde{X}\rangle_{T=0} = \sqrt{3}(\zeta - 1) + O(1 - \zeta)^2 \quad (11.77)$$

where $\zeta = 1 - [\frac{g-1}{3(4\pi)2}]\ \Lambda^2/m_{Pl}^2$ and β_0 is renormalized such that the energy at the minimum is zero. We have $\beta_0 = (2-\sqrt{3})\ \zeta + O(1-\zeta)^2$. In the same approximation, the finite temperature minimum of the free energy is given by

$$\langle\tilde{X}\rangle_T = \langle\tilde{X}\rangle_{T=0} - \frac{1}{24}(g-1)\ T^2/m_{Pl}^2. \quad (11.78)$$

And the energy in the potential is given by

$$V^T(\langle\tilde{X}\rangle_T) = 23\ \mu^4\ \exp(\tilde{X}^2)\ \frac{1}{24}(g-1)\ T^2/m_{Pl}^2. \quad (11.79)$$

It is thus reasonable to expect that the initial value of the Polonyi field at temperatures of order M_{Pl} is shifted away from its zero temperature minimum by an amount of order M_{Pl}.

The Polonyi field X has mass $m_X \simeq \mu^2/m_{Pl}$, and a decay rate into MSSM matter, $\Gamma_X \simeq m_X^3/m_{pl}^2$. Hence, like the gravitino, the Polonyi field (and generically all moduli) is long lived. Assuming the universe goes through a period of inflation at temperatures of order the GUT scale, the Polonyi field will typically obtain an additional contribution to the scalar potential proportional to $H_{inf}^2\ X^2$ which will have the effect of moving the Polonyi field further away from its zero temperature minimum.

The evolution of the Polonyi field and radiation in the early universe is determined by the equations[6]

$$\begin{aligned} \dot{\rho_X} + 3\,H\,\dot{X}^2 &= -\Gamma_X\,\dot{X}^2 \\ \dot{\rho}_r + 4\,H\,\rho_r &= +\Gamma_X\,\dot{X}^2 \\ H^2 &= \frac{\rho_X + \rho_r}{3\,m_{Pl}^2}. \end{aligned} \tag{11.80}$$

During the inflationary epoch, we have $H \gg \Gamma_X$ and the Polonyi field is overdamped and loses little energy. After reheating and at temperatures of order μ the Polonyi field begins to oscillate about its zero temperature minimum. At this point the kinetic and potential energy in the Polonyi field are approximately equal, and thus the Polonyi field acts like matter with zero pressure. At a temperature $T \leq \mu$ the energy density in the Polonyi field is of order $\rho_X \sim \mu\, T^3$ and it dominates the energy density of the universe. Then at a much lower temperature,

$$T_{decay} \sim (3\,m_{Pl}^2\,\Gamma_X^2/\mu)^{1/3}, \tag{11.81}$$

the Polonyi field decays and the universe reheats to a temperature, T_{reheat} satisfying energy conservation, i.e.

$$\rho_X(T_{decay}) = \rho_r(T_{reheat}). \tag{11.82}$$

We have

$$T_{reheat} = (g(T_{reheat})\pi^2/30)^{-1/4}(3\,m_{Pl}^2\,\Gamma_X^2)^{1/4}. \tag{11.83}$$

The entropy produced when the Polonyi field decays is huge. We have

$$\Delta = (\frac{T_{reheat}}{T_{decay}})^3 \sim (\frac{\mu^4}{3\,m_{Pl}^2\Gamma_X^2})^{1/4}. \tag{11.84}$$

Now let's put in some numbers. Given $m_X = \sqrt{3}m_{3/2}$ and with $m_{3/2} = 100\,\text{GeV}$ we find $m_X = 173\,\text{GeV}$. Then $\Gamma_x = 9 \times 10^{-31}(\frac{m_{3/2}}{100\,\text{GeV}})^3$ GeV. We have

$$T_{reheat} \sim 3\,m_{3/2}\,(\frac{m_{3/2}}{m_{Pl}})^{1/2} \sim 2\,\text{keV}\,(\frac{m_{3/2}}{100\,\text{GeV}})^{3/2} \tag{11.85}$$

and

$$\Delta \sim \frac{m_{Pl}}{3^{3/4}m_{3/2}} \sim 10^{16}\,(\frac{100\,\text{GeV}}{m_{3/2}}). \tag{11.86}$$

[6]See problem 9.

This is a problem unless $T_{reheat} \geq 7\,\text{MeV}$, i.e. temperatures above the nucleosynthesis scale, which requires

$$m_X \geq 40\,\text{TeV}, \quad m_{3/2} \geq 23\,\text{TeV}. \tag{11.87}$$

Therefore we see that a simultaneous solution to the gravitino problem (with heavy gravitinos) and the moduli problem is obtained with

$$m_{3/2} \geq 40\,\text{TeV}. \tag{11.88}$$

Note, while inflation can ameliorate the gravitino problem, it is unlikely to resolve the moduli problem. In addition, the baryon asymmetry will be diluted when the universe reheats. This is another serious problem requiring a solution.

11.2.5 Summary

The gravitino problem (with heavy gravitinos) can be solved by inflation, however the reheat temperature after inflation must be less than about $10^{10}\,\text{GeV}$ for heavy gravitinos with mass greater than 40 TeV. For gravitino LSPs, the reheat temperature after inflation is again less than 10^8–$10^9\,\text{GeV}$. Both cases require baryogenesis to occur at temperatures below the reheat temperature. Finally, the cosmological moduli problem and the heavy gravitino problem are solved with moduli and gravitino masses greater than $\sim$40 TeV, without inflation. But in either case, a great deal of entropy is released at very low temperatures which would invariably dilute the primordial baryon asymmetry. This is a serious problem. There are now several possible solutions.

1. All moduli obtain supersymmetric masses at a high scale. It has been shown that moduli stabilization at a high energy scale with supersymmetry breaking can solve the cosmological moduli problem [274]. It has also been suggested that a nilpotent Polonyi field can solve the cosmological Polonyi problem [275]. Thus we can have baryogenesis via leptogenesis after inflation. We would first attempt incorporating a solution such as this. However, these may not be sufficient to solve the generic cosmological moduli problem of string theories which typically have many moduli.[7]
2. Moduli are driven to the minimum of their potentials during inflation, i.e. so-called moduli trapping at points of enhanced symmetry [270, 277–280]. This may work with certain moduli, but it is unlikely to work for the dilaton in string theory. To solve this problem, we revert to solution (1). Such a solution may also be consistent with a GUT scale leptogenesis scenario.

[7]In particular Heterotic string models with a discrete $\mathbb{Z}_4^R$ symmetry it has been argued that most of the moduli may be stabilized in supersymmetric vacua with string scale masses [276].

In the latter two cases, low temperature baryogenesis mechanisms must be incorporated.

3. Thermal inflation above the weak scale (diluting moduli) followed by baryogenesis [281].
4. Moduli dominate the energy density during BBN, then decay and reheat above 10 MeV. Then it is possible to have baryogenesis via moduli decay [282].

See [272] for a recent discussion of this issue.

11.3 Dark Matter

The Standard Model of Cosmology ΛCDM includes about 23% by mass of some form of cold dark matter [246]. This component of the energy budget of the universe is necessary to explain the light to mass ratio of galactic clusters; the velocity curves of spiral galaxies or even the Baryon Acoustic Oscillations which are observed in the cosmic microwave background data (for a recent review, see [283] or the book by Baumann and McAllister [245]). Supersymmetric theories provide an abundance of possible dark matter candidates. If the theory has an R parity, then the lightest supersymmetric particle [LSP] is stable. Dark matter should be electromagnetically neutral. The possible LSP candidates of the MSSM include the lightest neutralino or the gravitino. In extensions of the MSSM, such as those which solve the strong CP problem via a Peccei-Quinn symmetry and an invisible axion, some combination of the axion and/or axino can also be dark matter.

In the former case, it was found that a well tempered dark matter candidate, i.e. part bino, wino, Higgsino can be consistent with Planck data (see [284]). Moreover, it was shown that well-tempered dark matter is obtainable with “mirage-mediation” boundary conditions [196]. In the latter case a detailed review of axino cold dark matter can be found in [285]. Also see [286, 287] for recent work on mixed axion/neutralino dark matter.

There are many on-going searches for dark matter, including direct searches in underground detectors; indirect searches looking for the annihilation products of dark matter particles in the galaxy or the production of dark matter particles in collider experiments. There are also specialized experiments searching for axionic dark matter. To date there has not been any observation.

Chapter 12
The Little Hierarchy Problem or Fine-Tuning

Supersymmetric theories were invoked to solve the gauge hierarchy problem, i.e. to explain why the weak scale is so much smaller than the Planck or GUT scales. It does this by extending the chiral symmetry of fermions to their bosonic superpartners. Without supersymmetry one must fine-tune the Higgs mass squared by one part in 10^{28}. But the question in the literature is how much fine-tuning is acceptable. The answer to this question is clearly subjective. Is one part in 100 too much or is one part in 1000 acceptable? Either way, the fine-tuning to 1 part in 10^{28} is avoided.

But the real question, which is of paramount importance to everyone, is whether or not supersymmetry can be discovered at the LHC. If all SUSY particles were much heavier than 10–100 TeV, their discovery is very unlikely with the present machine. With all of this said, I also believe it would be very important to find versions of SUSY which exhibit minimal fine-tuning; as long as they remain aesthetic. For example, it has been argued that the General Non-Minimal Supersymmetric Standard Model [GNMSSM] [288] has fine-tuning to less than 1 part in 20. In this section, let me focus on the fine-tuning found in the predictive $SO(10)$ model with Yukawa unification [146].

Consider the standard definition of fine-tuning. The question which is asked is how sensitive are low energy observables, in particular the Z mass, to variations of the fundamental parameters in the theory. As a warm up, prior to discussing fine-tuning in SUSY, let's ask how much fine-tuning is present in the Standard Model. For example, it is well known that the nucleon mass is determined dynamically via strong QCD interactions. In particular, the quarks obtain dynamical masses of order the renormalization group invariant QCD scale, $\Lambda_{QCD} \sim 300$ MeV. The nucleon mass is of order $3 \times \Lambda_{QCD}$, while the ρ meson mass is about $2 \times \Lambda_{QCD}$. One should note that the proton mass is determined to be much smaller than the Planck scale due to the logarithmic running of the strong coupling constant and it is accepted that

S. Raby, *Supersymmetric Grand Unified Theories*, Lecture Notes in Physics 939,
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the proton is "naturally" lighter than the Planck scale. We have

$$\Lambda_{QCD} = \exp(-8\pi^2/b_3 g_3^2)\, M_{Pl} \tag{12.1}$$

where Λ_{QCD} is evaluated at one loop. It determines the dynamical quark mass and hence the proton mass. We can define the fine-tuning parameter[1]

$$\begin{aligned}\Delta &= \frac{\partial \log \Lambda_{QCD}^2}{\partial \log g_3^2} = 2\,\frac{g_3^2}{\Lambda_{QCD}}\frac{\partial \Lambda_{QCD}}{\partial g_3^2} \\ &= 2\,\frac{g_3^2}{\Lambda_{QCD}}(\Lambda_{QCD}\,\frac{+8\pi^2}{b_3 g_3^4}) = \frac{16\pi^2}{b_3 g_3^2} \simeq \frac{4\pi}{b_0 \alpha_3(M_{Pl})} \sim 100.\end{aligned} \tag{12.2}$$

So perhaps we should conclude that fine-tuning to one part in 100 is natural.

We studied the fine-tuning of our $SO(10)$ model using the fine-tuning measure introduced by Ellis et al. [289], and studied in detail by Barbieri and Giudice [290],

$$\Delta_{\mathrm{BG}} = \max \Delta_{a_i}\,, \qquad \Delta_{a_i} = \left|\frac{\partial \ln M_Z^2}{\partial \ln a_i^2}\right|\,, \tag{12.3}$$

where a_is are input parameters of the model. This fine-tuning measure calculates the sensitivity of M_Z due to a small variation of the input parameters defined at the GUT scale.

Electroweak symmetry is broken radiatively in our model. From radiative electroweak symmetry breaking, the CP-odd Higgs mass, m_A, and the μ-term are calculated at one-loop [188]. This calculation requires the physical Z pole mass, M_Z. Hence, in our model, M_Z is fit precisely. To make sure that radiative electroweak symmetry breaking is consistent, m_A and μ are calculated iteratively until they converge.

On the other hand, when we calculate fine-tuning using Eq. (12.3), we use the benchmark points. The benchmark points are the inputs that produce minimum χ^2 value for the respective values of m_{16} and $M_{1/2}$. Hence, at each benchmark point, radiative electroweak symmetry breaking is consistent. Thus, instead of fixing M_Z and calculating m_A and μ iteratively, we then use the value of m_A and μ to calculate a new value of M_Z given new input parameters. We then compare this value of M_Z to the exact value to obtain the fine tuning parameter Δ_{BG}.

The input parameters that we vary are $a_i = \{\mu, M_{1/2}, m_{16}, m_{H_u}, m_{H_d}, A_0\}$. The results of our calculations are summarized in Table 12.1. From Table 12.1, we see that if there are no constraints on the input parameters (first five rows), then the fine-tuning is about 1 part in 10^5. However, if the GUT scale parameters are constrained such that $m_{H_{u,d}}/m_{16} \approx \sqrt{2}$ and $A_0/m_{16} \approx -2$, then the fine-tuning of our theory is reduced to about 1 part in 700 for $m_{16} = 25$ TeV. What does this mean?

[1]We use Eq. (5.44) with $\alpha_3(M_Z) = 0.1$, $b_0 = 7$ and M_G replaced by M_{Pl}.

Table 12.1 Without fixing any ratios, the fine-tuning is 1 part in 10^5

Varying parameters	m_{16} 10 TeV	15 TeV	20 TeV	25 TeV	30 TeV
Fine-tuning of benchmark points with $\alpha = 0$ and $M_{\tilde{g}} \approx 1.2$ TeV					
μ	140	190	210	360	490
$M_{1/2}$	260	340	400	430	450
m_{16}	12,000	27,000	47,000	74,000	110,000
m_{H_d}	760	1500	3900	6100	8700
m_{H_u}	10,000	23,000	40,000	62,000	89,000
A_0	9300	21,000	39,000	61,000	85,000
m_{16} with A_0/m_{16} fixed	22,000	49,000	87,000	130,000	190,000
m_{16} with $m_{H_u,d}/m_{16}$ fixed	9500	22,000	40,000	62,000	86,000
m_{16} with $m_{H_{u,d}}/m_{16}, A_0/m_{16}$ fixed	240	400	630	740	850
Fine-tuning of benchmark points with $\alpha = 1.5$ and $M_{\tilde{g}} \approx 1.2$ TeV					
μ	110	240	290	340	380
$M_{1/2}$	320	420	500	560	580
m_{16}	12,000	28,000	48,000	74,000	110,000
m_{H_d}	750	1500	4600	6000	8500
m_{H_u}	10,000	23,000	39,000	62,000	89,000
A_0	9200	21,000	39,000	60,000	86,000
m_{16} with A_0/m_{16} fixed	22,000	49,000	87,000	130,000	190,000
m_{16} with $m_{H_u,d}/m_{16}$ fixed	9600	21,000	39,000	61,000	87,000
m_{16} with $m_{H_{u,d}}/m_{16}, A_0/m_{16}$ fixed	330	450	670	890	1100

When the ratio of $m_{H_{u,d}}/m_{16}$ and A_0/m_{16} are fixed, the fine-tuning according to Eq. (12.3) is about 1 part in 700 for $m_{16} = 25$ TeV. Hence, from naturalness, we can infer that these ratios should be fixed in a more fundamental theory. In addition, fine-tuning increases as m_{16} increases. Hence in our model, as might be expected, smaller m_{16} is favored by naturalness

This suggests that, in a more fundamental and natural theory, the ratio of m_{16} with $m_{H_{u,d}}$ and A_0 could be fixed by constrained boundary conditions at the fundamental scale. Hence, one should combine these quantities before calculating fine-tuning. In Sect. 24.3.2 we consider how such special boundary conditions may be obtained in a string context.

Summary

We evaluated the amount of high scale fine-tuning of our model. In general we find fine-tuning of order 1 part in 10^5. However we note that with particular boundary conditions at the GUT scale (when the ratio of m_{16} to A_0 and $m_{H_{u,d}}$ are fixed at $A_0/m_{16} \approx -2$ and $m_{H_{u,d}}/m_{16} \approx \sqrt{2}$) the fine-tuning is reduced to 1 part in 700 for $m_{16} = 25$ TeV. We do not have a fundamental theory that gives these two ratios naturally. Nevertheless, in such a fundamental theory the amount of fine-tuning is reduced considerably.

Chapter 13
Problems of 4D GUTs

There are two aesthetic (perhaps more fundamental) problems concerning 4D GUTs. They have to do with the complicated sectors necessary for GUT symmetry breaking and Higgs doublet-triplet splitting. These sectors are sufficiently complicated that it is difficult to imagine that they may be derived from a more fundamental theory, such as string theory. In fact, in the heterotic string it is well-known that the largest chiral matter representation in either $SU(5)$ or $SO(10)$, which is massless in 4D, is the adjoint representation. Moreover for $SO(10)$ there are no examples of semi-realistic models with a massless adjoint. Why should you care about string theory? Because it is the only known possible extension of the Standard Model which includes quantum gravity. In order to resolve these difficulties, it becomes natural to discuss grand unified theories in higher spatial dimensions. These are the so-called orbifold GUT theories discussed in the next section. They have their own problems, i.e. they are non-renormalizable theories requiring an infinite number of higher dimensional counter terms. In later chapters we shall discuss ways of embedding orbifold GUTs into the Heterotic string.

Consider, for example, one of the simplest constructions in $SO(10)$ which accomplishes both tasks of GUT symmetry breaking and Higgs doublet-triplet splitting [149]. Let there be a single adjoint field, A, and *two* pairs of spinors, $C + \overline{C}$ and $C' + \overline{C}'$. The complete Higgs superpotential is assumed to have the form [148, 149]

$$\mathscr{W} = \mathscr{W}_A + \mathscr{W}_{ACC'} + (HAH' + SH'^2). \tag{13.1}$$

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The precise form of $\mathscr{W}_A$ does not matter, as long as $\mathscr{W}_A$ gives $\langle A \rangle$ the Dimopoulos-Wilczek form. $\mathscr{W}_{ACC'}$ makes the VEVs of C and $\overline{C}$ point in the $SU(5)$-singlet direction. The last term is the Higgs doublet-triplet splitting sector. The crucial term that couples the spinor and adjoint sectors together has the form

$$\mathscr{W}_{ACC'} = C \left(\frac{a_1}{M_*} Z A + \frac{b_1}{M_*} C \bar{C} + c_1 S \right) \bar{C}' + C' \left(\frac{a_2}{M_*} Z A + \frac{b_2}{M_*} C \bar{C} + c_2 S \right) \bar{C} \tag{13.2}$$

where Z and S are singlets. The critical point is that the VEVs of the primed spinor fields will vanish, and therefore the terms in Eq. (13.2) will not make a destabilizing contribution to $-F_A^* = \partial W / \partial A$. This is the essence of the mechanism.

In $SU(5)$ the construction which gives natural Higgs doublet-triplet splitting requires the $SU(5)$ representations **75**, **50**, $\overline{\mathbf{50}}$ and a superpotential of the form [73, 97]

$$\mathscr{W} \supset 75^3 + M 75^2 + 5_H\, 75\, 50 + \bar{5}_H\, 75\, \overline{50} + 50\, \overline{50}\, X. \tag{13.3}$$

Summary

SUSY GUTs in 4 space-time dimensions require both complicated GUT symmetry breaking sectors and Doublet-Triplet Higgs splitting. These sectors are unlikely to be derivable from string theory. Moreover, the solution to both problems are greatly simplified in string constructions. As a first step to string constructions, we consider SUSY GUTs in extra spatial dimensions in the next chapter, i.e. so called “Orbifold GUTs.”

Chapter 14
Orbifold GUTs

14.1 GUTs on a Circle

Let us briefly review the geometric picture of GUT models compactified on a circle S^1. The circle $S^1 \equiv \mathbb{R}^1/\mathscr{T}$ where $\mathscr{T}$ is the action of translations by $2\pi R$. All fields Φ are thus periodic functions of the fifth dimension $x_5 = y$ with

$$\Phi(x_\mu, y) \to \Phi(x_\mu, y + 2\pi R) = \Phi(x_\mu, y) \tag{14.1}$$

(see Fig. 14.1).

As the first example of an orbifold GUT consider a toy model, i.e. a pure $SO(3)$ gauge theory in five dimensions [291]. The gauge field is

$$A_M \equiv A_M^a\, T^a, \quad a = 1, 2, 3; \quad M, N = \{0, 1, 2, 3, 5\}. \tag{14.2}$$

The gauge field strength is given by

$$F_{MN} \equiv F_{MN}^a\, T^a = \partial_M A_N - \partial_N A_M + i[A_M, A_N] \tag{14.3}$$

where T^a are $SO(3)$ generators. The Lagrangian is

$$\mathscr{L}_5 = -\frac{1}{4 g_5^2 k}\, Tr(F_{MN} F^{MN}) \tag{14.4}$$

and we have $Tr(T^a\; T^b) \equiv k\delta^{ab}$. The inverse gauge coupling squared has mass dimensions one.

Let us first compactify the theory on $\mathscr{M}_4 \times S^1$ with coordinates $\{x_\mu, y\}$ and $y = [0, 2\pi R)$. The theory is invariant under the local gauge transformation

$$A_M(x_\mu, y) \to U\, A_M(x_\mu, y)\, U^\dagger - iU\, \partial_M\, U^\dagger, \quad U = \exp(i\theta^a(x_\mu, y)\, T^a). \tag{14.5}$$

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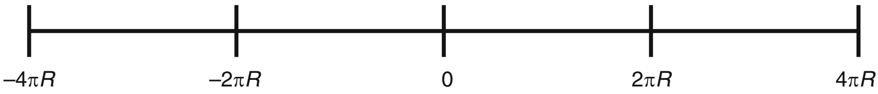

Fig. 14.1 The real line modded out by the space group of translations, $\mathscr{T}$

Consider the possibility $\partial_5 A_\mu \equiv 0$. We have

$$F_{\mu 5} = \partial_\mu A_5 + i[A_\mu, A_5] \equiv D_\mu A_5. \tag{14.6}$$

We can then define

$$\tilde{\Phi} \equiv A_5 \frac{\sqrt{2\pi R}}{g_5} \equiv A_5/g \tag{14.7}$$

where $g_5 \equiv \sqrt{2\pi R}\ g$ and g is the dimensionless 4D gauge coupling. The 5D Lagrangian reduces to the Lagrangian for a 4D $SO(3)$ gauge theory with massless scalar matter in the adjoint representation, i.e.

$$\mathscr{L}_5 = \frac{1}{2\pi R}[-\frac{1}{4g^2k}\ Tr(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2k}\ Tr(D_\mu \tilde{\Phi}\ D^\mu \tilde{\Phi})]. \tag{14.8}$$

This resembles the Georgi-Glashow model [292] of an $SO(3)$ gauge theory interacting with an isovector Higgs field. There are two differences, however. First, there is no potential $V(\tilde{\Phi}) = \lambda(\tilde{\Phi}^a\tilde{\Phi}^a - V^2)^2$ for the Higgs field which would break the gauge symmetry down to $U(1)$ and second, the Higgs field depends on the 5th coordinate.[1] Although this analysis is limited to gauge fields satisfying $\partial_5 A_\mu = 0$, it nevertheless inspires the following discussion of symmetry breaking via Wilson loops. In general, however, $\partial_5 A_\mu \neq 0$ and we need to keep the full $Tr(F_{\mu 5}^2)$ term.

We can always choose a gauge such that $\partial^M A_M = 0$. In this gauge the free field equations of motion are given by $\partial_M \partial^M A_N = 0$. In general we have the mode expansion

$$A_M(x_\mu, y) = \sum_n [a_M^n \cos n\frac{y}{R} + b_M^n \sin n\frac{y}{R}] \tag{14.9}$$

where only the cosine modes with $n = 0$ have zero mass. Otherwise the 5D Laplacian $\partial_M \partial^M = \partial_\mu \partial^\mu + \partial_y \partial^y$ leads to Kaluza-Klein [KK] modes with effective 4D mass

$$m_n^2 = \frac{n^2}{R^2}. \tag{14.10}$$

[1] In [291] we constructed 't Hooft–Polyakov monopole strings in the 5D orbifold GUT theory.

Quantizing Non-Abelian Gauge Theories

Consider the five dimensional action of a pure $SU(N)$ gauge theory

$$S_{gauge} = -\frac{1}{2g_5^2} \int d^5x Tr \left[F_{MN} F^{MN} \right] \tag{14.11}$$

with normalization $Tr(T^a T^b) = \frac{1}{2}\delta_{ab}$ and gauge covariant derivative $D_M = \partial_M + iA_M$. Note, the gauge field strength is defined by $F_{MN} = \partial_M A_N - \partial_N A_M + i[A_M, A_N] \equiv -i[D_M, D_N]$.

We shall consider working in the background field gauge defined by expanding about a solution to the classical equations of motion, i.e. $D^{M\,cl} F^{cl}_{MN} = 0$. The gauge field is then given by $A_M = A^{cl}_M + a_M$ where a_M is the quantum field. We find

$$F_{MN} = F^{cl}_{MN} + D^{cl}_M a_N - D^{cl}_N a_M + i[a_M, a_N] \quad \text{with} \quad {D_M}^{cl} = \partial_M + iA^{cl}_M \tag{14.12}$$

and thus

$$\begin{aligned} S_{gauge} = &-\frac{1}{2g_5^2} \int d^5x Tr \left[F^{cl}_{MN} F^{MN\,cl} - 2(D^{cl}_M F^{cl}_{MN}) a_N \right] \\ &+ \frac{1}{g_5^2} \int d^5x Tr \left[-(D^{cl}_M a_N)\ (D^{M\,cl} a^N) + (D^{cl}_M a_N)\ (D^{N\,cl} a^M) - i\, F^{cl}_{MN}\ [a_M, a_N] \right] \\ &+ \ \text{cubic and quartic terms in} \ \ a_M . \end{aligned} \tag{14.13}$$

Note the second term in the first line vanishes by the equations of motion. Finally we use

$$\begin{aligned} \int d^5x Tr \left[(D^{cl}_M a_N)\ (D^{N\,cl} a^M) \right] &= \int d^5x Tr \left[-a_N\ (D^{cl}_M\ D^{N\,cl}\ a^M) \right] \\ &= \int d^5x Tr \left[a_N\ [D^{N\,cl}, {D_M}^{cl}]\ a^M) - a_N\ (D^{N\,cl}\ {D_M}^{cl}\ a^M) \right] \\ &= \int d^5x Tr \left[i\, a^N\ F^{cl}_{NM}\ a^M) + (D^{N\,cl} a_N)\ (D^{M\,cl} a_M) \right] . \end{aligned} \tag{14.14}$$

Hence we find

$$\begin{aligned} S_{gauge} = &-\frac{1}{2g_5^2} \int d^5x Tr \left[F^{cl}_{MN} F^{MN\,cl} \right] \\ &+ \frac{1}{g_5^2} \int d^5x Tr \left[(a_N\ (D^{M\,cl}\ {D_M}^{cl}\ a^N) + (D^{M\,cl} a_M)^2 + 3i\, a^M\ F^{cl}_{MN}\ a^N \right] \\ &+ \ \text{cubic and quartic terms in} \ \ a_M . \end{aligned} \tag{14.15}$$

We must necessarily choose a gauge fixing action, S_{gf}. We take the gauge fixing action

$$S_{gf} = -\frac{1}{g_5^2 \xi} \int d^5x Tr \left[(D^{M^{cl}} a_M)^2 \right]. \tag{14.16}$$

Hence

$$\begin{aligned} S_{gauge} + S_{gf} = & -\frac{1}{2g_5^2} \int d^5x Tr \left[F^{cl}_{MN} F^{MN^{cl}} \right] \\ & + \frac{1}{g_5^2} \int d^5x Tr \left[(a_N \; (D^{M^{cl}} \; D_M{}^{cl} \; a^N) + (1 - \frac{1}{\xi})(D^{M^{cl}} a_M)^2 + 3i \; a^M \; F^{cl}_{MN} \; a^N \right] \\ & + \text{ cubic and quartic terms in } \; a_M. \end{aligned} \tag{14.17}$$

The Fadeev-Popov ghost action in this case is given by

$$S_{FP} = 2\frac{1}{g_5^2} \int d^5x Tr \left[(D_M{}^{cl} \; \bar{c}) \; (D^{M^{cl}} \; c) + i \; [a_M, \bar{c}] \; D^{M^{cl}} \; c \right]. \tag{14.18}$$

Gauge invariance in the background field gauge is a bit different than usual. In the background field gauge the classical solution satisfies the same transformation law as in Eq. (14.5). However the quantum fluctuation of the gauge field, a_M, satisfies the homogeneous gauge transformation

$$a_M(x_\mu, y) \to U \; a_M(x_\mu, y) \; U^\dagger. \tag{14.19}$$

In the next section we shall study the gauge theory with background gauge fields given by Wilson lines. In this case the gauge bundle is flat, i.e. $F^{cl}_{NM} = 0$ and the equations of motion for the quantum field a_M is given by

$$D^{N^{cl}} \; D^{cl}_N \; a_M = 0. \tag{14.20}$$

Wilson Loop Gauge Symmetry Breaking on $M_4 \otimes S^1$

Now, as before, we assume the 5th dimension is compactified on a circle S^1 parametrized by $y \in [0, 2\pi R)$. The gauge symmetry can then be broken by the presence of a background gauge field A_5. This symmetry breaking mechanism is known as Hosotani or Wilson line symmetry breaking [293–298]. Consider the constant background to be along the third isospin direction,

$$A_5(y) = A_5^3 T^3. \tag{14.21}$$

Using the single valued gauge transformation (periodic under $y \to y + 2\pi R$) given by Eq. (14.5) with $\theta(x_\mu, y) = -ny/R, n \in \mathbb{Z}$:

$$U(y) = \exp\left(-inT^3 \frac{y}{R}\right), \tag{14.22}$$

we obtain the transformation of A_5^3:

$$A_5^3 \to A_5^3 + n/R. \tag{14.23}$$

Therefore the gauge non-equivalent values of A_5^3 can be chosen to lie between 0 and $1/R$. The holonomy due to this constant background gauge field is given by

$$T = \exp\left(i \oint A_5 dy\right) = \exp\left(i\alpha T^3\right). \tag{14.24}$$

with the arbitrary parameter $\alpha \equiv 2\pi R A_5^3$. Note the set of possible holonomies $\{\mathbb{1}, T^{\pm 1}, T^{\pm 2}, \cdots\}$ provides a mapping of the gauge group into the discrete group $\mathbb{Z}$. This non-trivial holonomy affects the spectrum of the theory. A massless periodic scalar field ϕ (satisfying $\phi(y + 2\pi R) = \phi(y)$) with isospin eigenvalue I_3 can be decomposed into Kaluza-Klein modes

$$\phi_{(n)}(x_\mu) \exp(iny/R). \tag{14.25}$$

The 5-dimensional wave equation $D^M D_M \phi = 0$ splits into an infinite set of 4-dimensional wave equations for Kaluza-Klein modes $\phi_{(n)}$ with masses given by

$$m_{(n)}^2 \phi_{(n)} \exp(iny/R) = -\left(\partial_y + iA_5^3 T^3\right)^2 \phi_{(n)} \exp(iny/R) = \left(\frac{n}{R} + A_5^3 I_3\right)^2 \phi_{(n)} \exp(iny/R). \tag{14.26}$$

It is now easy to obtain the spectrum of gauge fields.[2] The gauge field $A_\mu^3(y)$ has $I_3 = 0$ and therefore its KK modes are not affected by the holonomy. The zero mode of this field corresponds to the gauge field of the unbroken $U(1)$. On the other hand, the masses of the KK modes of the $W^\pm$ gauge bosons, with $I_3 = \pm 1$, are given by $m_{(n)} = |\frac{n}{R} \pm A_5^3|$. If $A_5^3 \neq \frac{k}{R}$, where $k \in \mathbb{Z}$, the gauge bosons $W^\pm$ are all massive. Clearly the $SO(3)$ symmetry is broken to $U(1)$. Note, the symmetry breaking scale satisfies $0 \leq A_5^3 < 1/R$, but is otherwise unconstrained.

[2]Recall, for a background field gauge with $A_M = A_M^{cl} + a_M$ where A_M^{cl} is the background value of the gauge field and a_M are the small fluctuations, the background covariant derivative is given by $D_M^{cl} \equiv \partial_M + i[A_M^{cl}, \ ]$. If we use the covariant gauge fixing condition $D^{M^{cl}} a_M \equiv 0$, then the gauge field equations of motion are given by $D^{M^{cl}} D_M^{cl} a_N + 3iF_{NM}^{cl} a^M = 0$. Note, for a constant background gauge field $F_{NM}^{cl} \equiv 0$.

Gauge Picture with Vanishing Background

A constant background gauge field A_5^3 may be gauged away with the non-periodic gauge transformation

$$U(y) = \exp\left(iyA_5^3T^3\right). \tag{14.27}$$

In this gauge the covariant derivative in Eq. (14.26) is trivial, i.e. $D_5 = \partial_5$. Nevertheless it is easy to see that, as expected, the physics is unchanged.

This gauge transformation is not single valued and thus the periodicity condition $\phi(y + 2\pi R) = \phi(y)$ becomes for the gauge transformed fields $\phi'(y) = U(y)\,\phi(y)$

$$\phi'(y + 2\pi R) = \exp\left(i\alpha T^3\right)\phi'(y). \tag{14.28}$$

Now the mode expansions are of the form

$$\phi_{(n)}(x_\mu)\exp\left[i\left(n/R + A_5^3I_3\right)y\right] \tag{14.29}$$

resulting in the identical spectrum as before.

14.2 SO(3) Gauge Theory on $M_4 \otimes S^1/Z_2$

The S^1/Z_2 Orbifold

The S^1/Z_2 orbifold is a circle S^1 modded out by a Z_2 parity symmetry: $y \to -y$. The 5th dimension is now a line segment $y \in [0, \pi R]$. This orbifold has two fixed points at $y = 0$ and πR. The Lagrangian (14.4) is invariant under the parity transformation

$$A_\mu(-y) = A_\mu(y) \tag{14.30}$$

$$A_5(-y) = -A_5(y). \tag{14.31}$$

As in the case of compactification on a circle we consider a constant background for A_5^3 [Eq. (14.21)]. Clearly such a background is not consistent with the parity operation, Eq. (14.31). However, following [299] we define a generalized parity by combining the parity transformation (14.31) with the gauge transformation (14.23), for $n = 1$, $A_5^3 \to A_5^3 + 1/R$. We then look for a consistent solution with constant A_5^3. There are now only two possible values for A_5^3. The possibility $A_5^3 = 0$ is obviously allowed, but in this case the gauge symmetry is unbroken. The only nontrivial choice corresponds to $A_5^3(y) = \frac{1}{2R}$ which changes sign under the "naive" parity, $A_5^3(-y) = -\frac{1}{2R}$, but is gauge equivalent to its original value. Therefore, instead of (14.30)–(14.31) we define the fields for negative y, in the region $-\pi R <$

$y < 0$, in terms of the fields defined for positive y in the fundamental domain, $0 < y < \pi R$, via the generalized parity transformation (i.e. a combined "naive" parity transformation (14.31) and a gauge transformation) such that, in general:

$$A_\mu(-y) = U(-y)A_\mu(y)U^\dagger(-y) - iU(-y)\partial_\mu U^\dagger(-y), \tag{14.32}$$

$$A_5(-y) = -U(-y)A_5(y)U^\dagger(-y) - iU(-y)\partial_{-y}U^\dagger(-y), \tag{14.33}$$

with

$$U(y) = \exp\left(-i\frac{y}{R}T^3\right) \tag{14.34}$$

It is useful to define new fields, $W^\pm$, in a usual way from A^1 and A^2:

$$W^\pm = \frac{1}{\sqrt{2}}\left(A^1 \mp iA^2\right), \quad T^\pm = \frac{1}{\sqrt{2}}\left(T^1 \pm iT^2\right). \tag{14.35}$$

With this definition we have $A^1T^1 + A^2T^2 = W^+T^+ + W^-T^-$ and $[T^3, T^\pm] = \pm T^\pm$. Using the identity[3]

$$\exp\left(i\frac{y}{R}T^3\right)T^\pm \exp\left(-i\frac{y}{R}T^3\right) = \exp\left(\pm i\frac{y}{R}\right)T^\pm \tag{14.36}$$

it is easy to show that the generalized parity transformation acts on gauge fields as follows:

$$W_\mu^\pm(-y) = \exp\left(\pm i\frac{y}{R}\right)W_\mu^\pm(y), \tag{14.37}$$

$$W_5^\pm(-y) = -\exp\left(\pm i\frac{y}{R}\right)W_5^\pm(y), \tag{14.38}$$

$$A_\mu^3(-y) = A_\mu^3(y), \tag{14.39}$$

$$A_5^3(-y) = -A_5^3(y) + \frac{1}{R}. \tag{14.40}$$

To summarize, using a more compact notation, we have the following constraints on the fields (valid for all modes, except the constant piece of A_5^3). Under the generalized parity transformation the fields ϕ_P (with $P = \pm 1$) satisfy:

$$\phi_P(-y) = P\exp\left(i\frac{y}{R}I_3\right)\phi_P(y) \tag{14.41}$$

[3] See problem 10.

with isospin eigenvalue $I_3 = \pm 1, 0$. The periodicity condition is given by:

$$\phi_P(y + 2\pi R) = \phi_P(y). \tag{14.42}$$

We then obtain the following decomposition into KK modes:

$$\phi_+(x_\mu, y) = \sum_{n=0}^{\infty} \phi_+^{(n)}(x_\mu) \exp\left(-i\frac{y}{2R} I_3\right) \cos n\frac{y}{R} \quad \text{for even } I_3, \tag{14.43}$$

$$\phi_+(x_\mu, y) = \sum_{n=0}^{\infty} \phi_+^{(n)}(x_\mu) \exp\left(-i\frac{y}{2R} I_3\right) \cos(n + 1/2)\frac{y}{R} \quad \text{for odd } I_3, \tag{14.44}$$

$$\phi_-(x_\mu, y) = \sum_{n=0}^{\infty} \phi_-^{(n)}(x_\mu) \exp\left(-i\frac{y}{2R} I_3\right) \sin(n + 1)\frac{y}{R} \quad \text{for even } I_3, \tag{14.45}$$

$$\phi_-(x_\mu, y) = \sum_{n=0}^{\infty} \phi_-^{(n)}(x_\mu) \exp\left(-i\frac{y}{2R} I_3\right) \sin(n + 1/2)\frac{y}{R} \quad \text{for odd } I_3. \tag{14.46}$$

From transformations (14.37)–(14.40) we see that the KK mode expansion of A_μ^3 [(+) field with $I_3 = 0$] is given in Eq. (14.43) with corresponding masses n/R. This is the only field which has a zero mode. It corresponds to the gauge field of the unbroken $U(1)$. The expansion of $W_\mu^\pm$ [(+) field with $I_3 = \pm 1$] is given in Eq. (14.44) with corresponding masses $(n + 1/2)/R$. Similarly, the expansion of $W_5^\pm$ [(–) field with $I_3 = \pm 1$] is given in Eq. (14.46) with corresponding masses $(n + 1/2)/R$. And finally, the expansion of A_5^3 [(–) field with $I_3 = 0$] is given by Eq. (14.45) up to the value of the constant background:

$$A_5^3(x_\mu, y) = \frac{1}{2R} + \sum_{n=0}^{\infty} A_5^{3(n)}(x_\mu) \sin(n + 1)\frac{y}{R}. \tag{14.47}$$

The holonomy T in this case is given by

$$T = \exp(i \oint A_5^3 T^3) = \exp(i\pi T^3) = diag(-1, -1, 1). \tag{14.48}$$

Hence $T^2 = \mathbb{1}$ or the set of possible holonomies $\{\mathbb{1}, T\}$ maps the gauge group into the discrete group $\mathbb{Z}_2$. Unlike the case of Wilson loops on S^1 discussed in Sect. 14.1, the background gauge field and consequently the holonomy on S^1/Z_2 can only take discrete values.

Now let us consider the gauge picture with vanishing background gauge field. As in the case of compactification on a circle, we can gauge away the constant background by the non-single valued gauge transformation given in Eq. (14.27). The transformations under the generalized parity are now those of Eqs. (14.30) and (14.31). In addition the non-single valued gauge transformation changes the periodicity condition as in Eq. (14.28) with $\alpha = \pi$.

To obtain the spectrum of KK modes of a field ϕ we consider both the transformation under parity and the effect of a non-trivial holonomy. Under parity,

$$\mathscr{P}: \quad \phi_{PT}(y) \;\rightarrow\; \phi_{PT}(-y) = P\phi_{PT}(y), \tag{14.49}$$

with $P^2 = 1$ or $P = \pm 1$. When going around the circle, the fields transform in the following way:

$$\mathscr{T}: \quad \phi_{PT}(y) \;\rightarrow\; \phi_{PT}(y + 2\pi R) = T\phi_{PT}(y) \tag{14.50}$$

with $T^2 = \mathbb{1}$ or $T = \pm 1$. Therefore there are four different kinds of fields $\phi_{\pm\pm}$ corresponding to the four different combinations of (P, T). It is easy to see that a field with given (P, T) can be expanded into the following modes:

$$\begin{aligned}
\xi_n(+,+) &= \cos n\frac{y}{R} \\
\xi_n(+,-) &= \cos(n+1/2)\frac{y}{R} \\
\xi_n(-,+) &= \sin(n+1)\frac{y}{R} \\
\xi_n(-,-) &= \sin(n+1/2)\frac{y}{R}
\end{aligned} \tag{14.51}$$

Only the $(+,+)$ fields have massless zero modes. Of all the gauge fields only A^3_μ is a $(+,+)$ field with a zero mode. $W^\pm_\mu$, A^3_5 and $W^\pm_5$ are $(+,-)$, $(-,+)$ and $(-,-)$ fields, respectively. Clearly the mode expansion and the corresponding KK masses are the same as in the previous picture. Note, our gauge transformation parameters [Eq. (14.5)] are constrained to satisfy $\theta^3(x_\mu, y) = \theta^3_n(x_\mu)\xi_n(+,+)$ and $\theta^{1,2}(x_\mu, y) = \theta^{1,2}_n(x_\mu)\xi_n(+,-)$. Hence, $SO(3)$ is the symmetry everywhere in the five dimensions, EXCEPT on the boundary at $y = \pi R$.

Correspondence to $S^1/(Z_2 \times Z_2')$ Orbifold

The S^1/Z_2 orbifold with holonomy T in the gauge picture without a constant background gauge field is directly related to the $S^1/(Z_2 \times Z_2')$ orbifold used recently in the literature [298, 300–310]. This correspondence is also evident in the work of [299, 311]. We just need to identify the $S^1/(Z_2 \times Z_2')$ orbifold with S^1, a circle of circumference $4\pi R$, divided by the Z_2 transformation $y \to -y$ and Z_2' transformation $y' \to -y'$, where $y' \equiv y - \pi R$. The physical space is again the line segment $y \in [0, \pi R]$ with orbifold fixed points at $y = 0$ and πR. It is easy to see that $\mathscr{P}' \in Z_2'$ in this picture corresponds to the combined translation and parity transformation in the previous picture, namely $\mathscr{P}' = \mathscr{T}\mathscr{P}$. Note, a point at $y = y_0$ which corresponds to $y' = y_0 - \pi R$ is transformed by Z_2' into the point $y' = -(y_0 - \pi R)$ corresponding to $y = -y_0 + 2\pi R$; this is equivalent to the action of $\mathscr{T} Z_2$ on the point at $y = y_0$, see Fig. 14.2.

The action of Z_2 on the fields is given by

$$\mathscr{P}: \quad \phi_{PP'}(y) \;\to\; \phi_{PP'}(-y) = P\phi_{PP'}(y), \tag{14.52}$$

with $P^2 = 1$ or $P = \pm 1$. Similarly, under Z_2' we have

$$\mathscr{P}': \quad \phi_{PP'}(\pi R + y') \;\to\; \phi_{PP'}(\pi R - y') = P'\phi_{PP'}(\pi R + y') \tag{14.53}$$

with $P' = TP$ and $(P')^2 = \mathbb{1}$ or $P' = \pm 1$.

It is easy to see what the holonomy means in this picture. Since points y_0 and $y_0 + 2\pi R$ are identified, the closed loop corresponds to going around half of the circle (the circumference of the circle in this picture is $4\pi R$). Going around the whole circle (from y_0 to $y_0 + 2\pi R$ and then from $y_0 + 2\pi R$ to $y_0 + 4\pi R$) clearly corresponds to T^2. From Eq. (14.31) we see that going from $y_0 + 2\pi R$ to $y_0 + 4\pi R$ is equivalent to going backwards from $y_0 + 2\pi R$ to y_0. Therefore $T^2 = \mathbb{1}$ and there are only two possibilities for holonomy, $T = +1$ and $T = -1$, the same as in the S^1/Z_2 picture. Hence we have $T \in \mathbb{Z}_2$. Note, in the above we have assumed that P and T

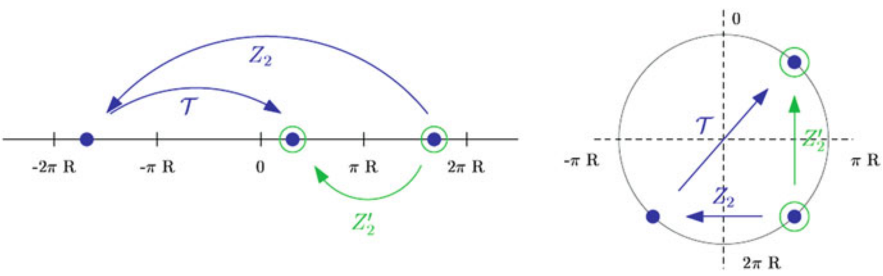

Fig. 14.2 The Z_2' parity transformation is equivalent to the combined Z_2 parity transformation and translation $\mathscr{T}$

can be simultaneously diagonalized. In general however P and T do not commute. In this case we would have $P\,T\,P = T^{-1}$.

14.3 GUTs on an Orbi-Circle: Brief Review

Let us briefly review the geometric picture of orbifold GUT models compactified on an orbi-circle $\mathrm{S}^1/\mathbb{Z}_2$. The circle $S^1 \equiv \mathbb{R}^1/\mathscr{T}$ where $\mathscr{T}$ is the action of translations by $2\pi R$. All fields Φ are thus periodic functions of y (up to a finite gauge transformation), i.e.

$$\mathscr{T} : \Phi(x_\mu, y) \to \Phi(x_\mu, y + 2\pi R) = T\,\Phi(x_\mu, y) \tag{14.54}$$

where $T \in \mathscr{G}$ satisfies $T^2 = 1$. This corresponds to the translation $\mathscr{T}$ being realized non-trivially by a degree-2 Wilson line (i.e., background gauge field—$\langle A_5\rangle \neq 0$ with $T \equiv \exp(i\oint\langle A_5\rangle dy)$). Hence the space group of $\mathrm{S}^1/\mathbb{Z}_2$ is composed of two actions, a translation, $\mathscr{T} : y \to y + 2\pi R$, and a space reversal, $\mathscr{P} : y \to -y$. There are two (conjugacy) classes of fixed points, $y = (2n)\pi R$ and $(2n+1)\pi R$, where $n \in \mathbb{Z}$ (Fig. 14.3).

The space group multiplication rules imply $\mathscr{T}\mathscr{P}\mathscr{T} = \mathscr{P}$, so we can replace the translation by a composite $\mathbb{Z}_2$ action $\mathscr{P}' = \mathscr{P}\mathscr{T} : y \to -y + 2\pi R$. The orbicircle $\mathrm{S}^1/\mathbb{Z}_2$ is equivalent to an $\mathbb{R}/(\mathbb{Z}_2 \times \mathbb{Z}_2')$ orbifold, whose fundamental domain is the interval $[0,\ \pi R]$, and the two ends $y = 0$ and $y = \pi R$ are fixed points of the $\mathbb{Z}_2$ and $\mathbb{Z}_2'$ actions respectively.

A generic 5D field Φ has the following transformation properties under the $\mathbb{Z}_2$ and $\mathbb{Z}_2'$ orbifoldings (the 4D space-time coordinates are suppressed),

$$\mathscr{P} : \Phi(y) \to \Phi(-y) = P\Phi(y)\,, \qquad \mathscr{P}' : \Phi(y) \to \Phi(-y + 2\pi R) = P'\Phi(y)\,, \tag{14.55}$$

where $P,\ P' \equiv PT = \pm$ are *orbifold parities* acting on the field Φ in the appropriate group representation.[4] The four combinations of orbifold parities give four types of

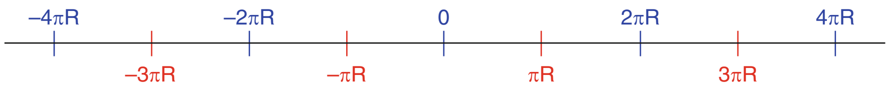

Fig. 14.3 The real line modded out by the space group of translations, $\mathscr{T}$, and a $\mathbb{Z}_2$ parity, $\mathscr{P}$

[4]Where it is assumed that $[P, T] = 0$.

states, with wavefunctions

$$\begin{aligned}
\zeta_m(++) &\sim \cos(my/R),\\
\zeta_m(+-) &\sim \cos[(2m+1)y/2R],\\
\zeta_m(-+) &\sim \sin[(2m+1)y/2R],\\
\zeta_m(--) &\sim \sin[(m+1)y/R],
\end{aligned} \tag{14.56}$$

where $m \in \mathbb{Z}$. The corresponding KK towers have masses

$$M_{\mathrm{KK}} = \begin{cases} m/R & \text{for } (PP') = (++)\,, \\ (2m+1)/2R & \text{for } (PP') = (+-) \text{ and } (-+)\,, \\ (m+1)/R & \text{for } (PP') = (--)\,. \end{cases} \tag{14.57}$$

Note that only the Φ_{++} field possesses a massless zero mode.

For example, consider the Wilson line $T = \exp(i\pi T^3) = \mathrm{diag}(-1,-1,1) \in SO(3)$. Let $A_\mu(y)$ $(A_5(y))$ have parities $P = +(-)$, respectively. Then only A^3_μ has orbifold parity $(++)$ and A^3_5 has orbifold parity $(--)$.[5] Define the fields

$$W^\pm = \frac{1}{\sqrt{2}}(A^1 \mp iA^2) \tag{14.58}$$

with $T^\pm = \frac{1}{\sqrt{2}}(T^1 \pm iT^2)$ and $[T^3, T^\pm] = \pm T^\pm$. Then $W^\pm_\mu$ $[W^\pm_5]$ have orbifold parity $(+-)$ $[(-+)]$, respectively. Thus the $SO(3)$ gauge group is broken to $SO(2) \approx U(1)$ in 4D. The local gauge parameters preserve the (P, T) parity/holonomy, i.e.

$$\begin{aligned}
\theta^3(x_\mu, y) &= \theta^3_m(x_\mu)\zeta_m(++)\\
\theta^{1,2}(x_\mu, y) &= \theta^{1,2}_m(x_\mu)\zeta_m(+-).
\end{aligned} \tag{14.59}$$

Therefore $SO(3)$ is *not* the symmetry at $y = \pi R$.

14.4 $SO(10)$ in 5D on $M_4 \otimes S^1/(\mathbb{Z}_2 \times \mathbb{Z}'_2)$

We have the adjoint representation of $SO(10)$ given by $A_M^{A=1,\cdots,45}$ with $M = 0, 1, 2, 3, 5$. Under $\mathbb{Z}_2$ we choose

$$\begin{aligned}
&A_\mu^{A=1,\cdots,45}, \quad P = +1 \\
&A_5^{A=1,\cdots,45}, \quad P = -1
\end{aligned} \tag{14.60}$$

[5] Note, $A^3_5(-y) = -A^3_5(y) + \frac{1}{R}$.

and under $\mathbb{Z}_2'$ we choose

$$\begin{aligned} A_\mu \in SO(6) \otimes SO(4) \equiv PS, \quad & P' = +1 \\ A_\mu \in SO(10)/PS, \quad & P' = -1 \\ A_5 \in PS, \quad & P' = -1 \\ A_5 \in SO(10)/PS, \quad & P' = +1. \end{aligned} \tag{14.61}$$

Recall the generators of $SO(10)$ in the ten dimensional representation given by

$$\Sigma_{ij}^{ab} = -\Sigma_{ij}^{ba}, \quad \text{with } \ a, b = 1, \cdots, 10; \ \ i, j = 1, \cdots, 10. \tag{14.62}$$

Then the gauge fields can be written as matrices with

$$(\tilde{A}_\mu)_{ij} = \frac{1}{2} A_\mu^{ab} \ \Sigma_{ij}^{ab}. \tag{14.63}$$

Under a global $SO(10)$ gauge transformation, $O_{ij} \in SO(10)$ satisfying $(O^T \ O) = \mathbb{1}$, the gauge fields transform by

$$(\tilde{A}'_\mu)_{ij} = (O \, \tilde{A}_\mu \ O^T)_{ij}. \tag{14.64}$$

Now choose the holonomy

$$T = diag(-1, -1, -1, -1, -1, -1, 1, 1, 1, 1), \quad \text{with} \quad T \in SO(10), \quad T^2 = \mathbb{1}. \tag{14.65}$$

This implies that

$$\mathscr{T} : \tilde{A}_{PT}(x_\mu, y) \to \tilde{A}_{PT}(x_\mu, y + 2\pi R) = T \, \tilde{A}_{PT}(x_\mu, y) \, T. \tag{14.66}$$

For

$$\begin{aligned} \tilde{A} \in PS \quad & \text{we have } \ T \, \tilde{A} \, T = \tilde{A} \\ \tilde{A} \in SO(10)/PS \ \ & \text{we have } \ T \, \tilde{A} \, T = -\tilde{A}. \end{aligned} \tag{14.67}$$

To summarize, under the parities $\{P, \ P'\}$ (where $P' = PT$) we have

$$A_{\mu_{(++)}} \in PS, \ \ A_{\mu_{(+-)}} \in SO(10)/PS, \ \ A_{5(--)} \in PS, \ \ A_{5(-+)} \in SO(10)/PS. \tag{14.68}$$

Hence only the gauge fields $A_\mu \in PS$ have massless modes. All other Kaluza-Klein modes have mass of order the compactification scale and larger.

Thus we can use this $S^1/\mathbb{Z}_2$ orbifold with non-trivial Wilson line to break the gauge group $SO(10)$ to the $SU(4)_C \times SU(2)_L \times SU(2)_R$ Pati-Salam group. Similarly, starting with $SU(5)$ and the holonomy $T \in SU(5)$ with $T = diag(1, 1, 1, -1, -1)$ we can use this orbifold to break the gauge group $SU(5)$ to the $SU(3)_C \times SU(2)_L \times U(1)_Y$ Standard Model gauge group.

14.5 Higgs Doublet-Triplet Splitting in Orbifold GUTs

Consider the gauge group $SO(10)$ with the Higgs doublets in the **10**. Under $SO(10)$ the Higgs transforms by

$$\mathbf{10}'_i = O_{ij}\, \mathbf{10}_j. \tag{14.69}$$

If we define the parities $\{P,\ T\}$ with

$$P : \mathbf{10}_i(x_\mu, y) \to \quad \mathbf{10}_i(x_\mu, -y) = P\, \mathbf{10}_i(x_\mu, y) \quad \text{with } P = +1 \tag{14.70}$$
$$T : \mathbf{10}_i(x_\mu, y) \to \mathbf{10}_i(x_\mu, y + 2\pi R) = T\, \mathbf{10}_i(x_\mu, y) \text{ with } T \text{ given in Eq. (14.65)},$$

then the color triplets, $C = (6, 1, 1)$ are, odd under T and the Higgs doublets, $H = (1, 2, 2)$ are even under T. To summarize, under the parities $\{P,\ P'\}$ we have

$$H_{(++)}, \quad C_{(+-)}. \tag{14.71}$$

Hence, only the Higgs doublets have massless modes.

Similarly for $SU(5)$, we can arrange it such that only the Higgs doublets have parity $(++)$ with massless modes.

14.6 Fermions in 5D

The Dirac algebra in 5D is given in terms of the 4×4 gamma matrices $\gamma_M,\ M = 0, 1, 2, 3, 5$ satisfying $\{\gamma_M, \gamma_N\} = 2g_{MN}$ with $\gamma_5^2 = -1$.[6] A four component massless Dirac spinor $\Psi(x_\mu, y)$ satisfies the Dirac equation

$$i\gamma_M \partial^M \Psi = 0 = i(\gamma_\mu \partial^\mu - \gamma_5 \partial_y)\Psi. \tag{14.72}$$

[6] The 5D $\gamma_5 = i \begin{pmatrix} -\mathbb{I}_{2\times 2} & 0 \\ 0 & \mathbb{I}_{2\times 2} \end{pmatrix}$.

In 4D the four component Dirac spinor decomposes into two Weyl spinors with

$$\Psi = \begin{pmatrix} \psi_1 \\ i\sigma_2 \psi_2^* \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{14.73}$$

where $\psi_{1,2}$ are two left-handed Weyl spinors. In general, we obtain the normal mode expansion for the fifth direction given by

$$\psi_{L,R} = \sum (a_n(x) \cos n\frac{y}{R} + b_n(x) \sin n\frac{y}{R}). \tag{14.74}$$

If we couple this 5D fermion to a local gauge theory, the theory is necessarily vector-like; coupling identically to both $\psi_{L,R}$.

We can obtain a chiral theory in 4D with the following parity operation

$$\mathscr{P} : \Psi(x_\mu, y) \to \Psi(x_\mu, -y) = P\Psi(x_\mu, y) \tag{14.75}$$

with $P = i\gamma_5$. We then have

$$\begin{aligned} \Psi_L &\sim \cos n\frac{y}{R} \\ \Psi_R &\sim \sin n\frac{y}{R}. \end{aligned} \tag{14.76}$$

Quarks and Leptons in 5D: SO(10)

Quarks and leptons are fermions and the SM gauge group is chiral. Therefore we must project out half of the 5D chiral modes as discussed above. We choose to retain only the left-handed chiral states. Now we need to discuss how the Wilson line affects the spectrum of fermions when they are placed into complete GUT multiplets. For example, consider quarks and leptons in the spinor representation of $SO(10)$. In Sects. 14.4 and 14.5 we used a Wilson line to break $SO(10)$ to PS and give mass to the Higgs triplets, keeping the doublets massless. Now we apply the same Wilson line to the **16**.

The holonomy

$$T = diag(-1,-1,-1,-1,-1,-1,1,1,1,1) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \otimes \mathbb{1}_{2\times 2} \tag{14.77}$$

on the ten dimensional representation. Note, that in this tensor product notation we have

$$B - L = \frac{2}{3}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \otimes \eta \tag{14.78}$$

with $\eta = \sigma_2$ [see Eq. (5.83)]. It is easy to show that[7]

$$T \equiv \exp(-i\,\frac{3}{2}\pi\ (B - L)). \tag{14.79}$$

Consider the field $\psi_L(x_\mu,\ y)$ in the spinor representation of $SO(10)$. Let

$$\mathscr{T} : \psi_L(x_\mu,\ y) \to \psi_L(x_\mu,\ y + 2\pi R) = T\ \psi_L(x_\mu,\ y). \tag{14.80}$$

Recall, the ψ_L contains the states $\psi_L \supset \{q,\ l;\ \bar{u},\ \bar{d},\ \bar{\nu},\ \bar{e}\}$. Moreover, under the action of T we have

$$T\ q = -i\ q;\ \ T\ l = -i\ l;\ \ T\ \bar{u} = i\ \bar{u};\ \ T\ \bar{d} = i\ \bar{d};\ \ T\ \bar{\nu} = i\ \bar{\nu};\ \ T\ \bar{e} = i\ \bar{e}. \tag{14.81}$$

Thus in terms of the Pati-Salam fields

$$\mathscr{Q} = (q\ l),\ \ \bar{\mathscr{Q}} = (\bar{q}\ \bar{l}) \tag{14.82}$$

where

$$\bar{q} = \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix},\ \bar{l} = \begin{pmatrix} \bar{\nu} \\ \bar{e} \end{pmatrix} \tag{14.83}$$

(see Sect. 5.1) we have

$$T\ \mathscr{Q} = -i\ \mathscr{Q};\ \ T\ \bar{\mathscr{Q}} = i\ \bar{\mathscr{Q}}. \tag{14.84}$$

Note, $T^2 = -1$, hence it is NOT a $\mathbb{Z}_2$ parity. Therefore we define a matter symmetry given by P_F, with $P_F \in \mathbb{Z}_4$ such that

$$P_F \psi_L = \pm i\ \psi_L. \tag{14.85}$$

Then the operator

$$P' = T\ P\ P_F \tag{14.86}$$

[7] See problem 11.

satisfied $P'^2 = 1$. We can thus choose either $\mathcal{Q}$ or $\bar{\mathcal{Q}}$ to have zero modes, BUT NOT BOTH. As a consequence, in order to obtain one complete family of quarks and leptons, we need two spinor representations, $\psi_{1L}, \quad \psi_{2L}$ in **16**s with opposite charges under P_F.

Quarks and Leptons in 5D: SU(5)

Quarks and leptons in $SU(5)$ sit in the

$$\mathbf{10} = \{\bar{u},\ q,\ \bar{e}\},\quad \bar{\mathbf{5}} = \{\bar{d},\ l\}. \tag{14.87}$$

The holonomy

$$T = \begin{pmatrix} +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = \exp(i\, 3\pi\ Y) \tag{14.88}$$

with

$$Y = \begin{pmatrix} -2/3 & 0 & 0 & 0 & 0 \\ 0 & -2/3 & 0 & 0 & 0 \\ 0 & 0 & -2/3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{14.89}$$

breaks $SU(5)$ to the SM gauge group and splits Higgs doublets and triplets. In order to obtain one complete family of quarks and leptons after orbifolding we need to, once again, double the 5D spectrum with two (**10**s and $\bar{\mathbf{5}}$s) such that we can identify the fields by the massless modes they contain, i.e.

$$\mathbf{10}(\bar{u}, \bar{e}),\ \mathbf{10}(q);\ \ \bar{\mathbf{5}}(l),\ \bar{\mathbf{5}}(\bar{d}). \tag{14.90}$$

This requires the addition of a matter parity defined by $P_F = \pm 1$, such that

$$P' = T\, P\, P_F. \tag{14.91}$$

Note, given that the Higgs doublets are contained in a **5** or $\bar{\mathbf{5}}$ of $SU(5)$, in order that the color triplets obtain odd charge under T we also need to add an additional matter parity $P_F = -1$ on the Higgs multiplet.

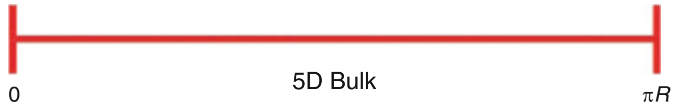

Fig. 14.4 The 5D orbifold is a line segment with fixed points at 0 and πR. At the fixed points the theory is four dimensional. We can also put states at the fixed points. In fact they do not need to be in complete multiplets of the GUT group at πR, because the GUT symmetry is broken at this fixed point. We only have PS for an $SO(10)$ GUT or the SM for an $SU(5)$ GUT

Quarks and Leptons on 4D Fixed Points

The orbifold in 5D is described by 4D Lorentz space $\times$ a line segment $[0, \ \pi R]$ (see Fig. 14.4). The GUT group is a symmetry in the 5D bulk, but it is broken to a subgroup on the 4D fixed point at πR. In an orbifold GUT field theory we have the option of putting states either in the bulk or on the 4D boundary surfaces at 0 and πR. Typically we prefer to put the gauge and Higgs fields in the bulk, since they couple to states in the bulk or on the boundary surfaces. However we have the option of putting the matter fields either in the bulk or on the boundary. The decision depends on the phenomenology we wish to obtain. We shall discuss this in more detail later when we consider SUSY GUTs on orbifolds.

14.7 Supersymmetric Orbifolds

We define the chiral superfields

$$H(y_\mu, \theta_\alpha) = \ h(y_\mu) + \sqrt{2}(\theta \tilde{h}(y_\mu)) + \theta^2 \ F_h(y_\mu) \tag{14.92}$$

$$H^c(y_\mu, \theta_\alpha) = h^c(y_\mu) + \sqrt{2}(\theta \tilde{h}^c(y_\mu)) + \theta^2 \ F_{h^c}(y_\mu) \tag{14.93}$$

where the fields, H^c, are charge conjugates of H. The fundamental chiral supermultiplet includes a complex scalar h and a Weyl spinor $\tilde{h}$. However, as we said earlier, the minimal spinor in 5D is a four component spinor. Hence the minimal supermultiplet in 5D, written in terms of 4D superfields, must include two complex scalars and 4 fermion field degrees of freedom. Thus we have two 4D superfields

$$H(y_\mu, y_5, \theta_\alpha) + H^c(y_\mu, y_5, \theta_\alpha) \tag{14.94}$$

with the component fields

$$h, \ h^{c*} \leftrightarrow \Psi_h = \begin{pmatrix} \tilde{h} \\ i\sigma_2 \ \tilde{h^c}^* \end{pmatrix}. \tag{14.95}$$

This actually corresponds to the degrees of freedom of an $N = 2$ SUSY hypermultiplet in 4D, which is an $N = 1$ supermultiplet in 5D.

Similarly for a gauge supermultiplet in 4D we have (in the adjoint representation of the group) the degrees of freedom of a massless gauge field, A_μ, (with 2 helicities) and a gaugino, λ, (a Weyl spinor with 2 helicities). Note the 2 helicities correspond to a representation of the "little group," $SO(2)$, for massless particles in 4D plus parity. In 5D the "little group" for massless particles is $SO(3)$ and therefore a vector has 3 degrees of freedom. Hence the vector supermultiplet in 5D includes 3 degrees of freedom for each vector field, A_M, 1 real scalar, ϕ, and a Dirac fermion, $\Psi_\lambda = \begin{pmatrix} \lambda_1 \\ i\sigma_2\, \lambda_2^* \end{pmatrix}$ where $\lambda_{1,2}$ are left-handed Weyl spinors. These states correspond to the vector multiplet of $N = 1$ SUSY in 5D. In terms of 4D massless superfields we have 2 bosonic and fermionic degrees of freedom in a 4D $N = 1$ vector multiplet

$$V \supset (A_\mu,\ \lambda_1) \tag{14.96}$$

and and a 4D chiral multiplet

$$\Sigma = (\frac{1}{\sqrt{2}}(\phi + iA_5),\ \lambda_2). \tag{14.97}$$

Again, we have $N = 1$ in 5D is equivalent to $N = 2$ in 4D. In fact we can see this given the SUSY generators in 5D,

$$Q_\alpha = \begin{pmatrix} Q_1 \\ i\sigma_2\, Q_2^* \end{pmatrix} = \begin{pmatrix} Q_L \\ Q_R \end{pmatrix}, \tag{14.98}$$

i.e. it's equivalent to two SUSY generators in 4D. For example, operating with Q_i on $\tilde{\Sigma} = \frac{1}{\sqrt{2}}(\phi + iA_5)$ we have $\lambda_i \Leftarrow Q_i\, \tilde{\Sigma}$ and $A_\mu \Leftarrow Q_i\, \lambda_j \Leftarrow Q_i\, Q_j\, \tilde{\Sigma}$. Similarly for the $N = 2$ hypermultiplet in 4D.

These results were formalized in [312]. They show in detail that the 5D SUSY Lagrangian can be written in terms of 4D superfields. The general result is given by

$$\begin{aligned} \mathscr{L} = & \int d^4\theta \frac{1}{kg_5^2}\, Tr[(\sqrt{2}\partial_5 + \Sigma^\dagger)\, e^{-V}\, (-\sqrt{2}\partial_5 + \Sigma)\, e^V + \partial_5\, e^{-V}\, \partial_5\, e^V] \\ & + \int d^2\theta \frac{1}{4kg_5^2}\, Tr[W^\alpha\, W_\alpha] + h.c. \\ & + \int d^4\theta [H^c\, e^V\, H^{c\dagger} + H^\dagger\, e^{-V}\, H] \\ & + \int d^2\theta (H^c\, (m + (\partial_5 - \frac{1}{\sqrt{2}}\, \Sigma))\, H) + h.c. \end{aligned} \tag{14.99}$$

with an action

$$S = \int d^5x\, \mathscr{L}. \tag{14.100}$$

Orbifolding 5D SUSY: A Simple Example

Define a $\mathbb{Z}_2$ orbifold with action P given by

$$\begin{aligned}
\mathscr{P}: \quad V(x_\mu,\ y,\ \theta) &\to V(x_\mu,\ -y,\ \theta) = V(x_\mu,\ y,\ \theta) \\
\Sigma(x_\mu,\ y,\ \theta) &\to \Sigma(x_\mu,\ -y,\ \theta) = -\Sigma(x_\mu,\ y,\ \theta) \\
H(x_\mu,\ y,\ \theta) &\to H(x_\mu,\ -y,\ \theta) = PH(x_\mu,\ y,\ \theta) \\
H^c(x_\mu,\ y,\ \theta) &\to H^c(x_\mu,\ -y,\ \theta) = -PH^c(x_\mu,\ y,\ \theta).
\end{aligned} \tag{14.101}$$

Only even parity fields, $V,\ H$ (or H^c) not both, have zero modes. Thus we have used this $\mathbb{Z}_2$ parity to break $N = 2$ SUSY to $N = 1$ SUSY in 4D. Note, Σ necessarily has odd parity, since under the orbifold, $\partial_5 \to -\partial_5$ and we can only orbifold by symmetries of the action. Note also, if there were a 5D mass term m, then we require m to depend on y such that under the orbifold $m(y) \to -m(-y)$,

As another simple example, we can write down the 5D orbifold version of SUSY QED. We have

$$\begin{aligned}
S = \int d^5x\{ &\frac{1}{e_5^2} \int d^4\theta[(\sqrt{2}\partial_5 + \Sigma^\dagger)\, e^{-V}\, (-\sqrt{2}\partial_5 + \Sigma)\, e^V + \partial_5\, e^{-V}\, \partial_5\, e^V] \\
&+ \frac{1}{4e_5^2} \int d^2\theta[W^\alpha\ W_\alpha] + h.c. \\
&+ \int d^4\theta([E^c\, e^V\, E^{\dagger c} + E^\dagger\, e^{-V}\, E] + [\bar{E}\, e^V\, \bar{E}^\dagger + \bar{E}^{c\dagger}\, e^{-V}\, \bar{E}^c]) \\
&+ \int d^2\theta(E^c\, (m_1 + (\partial_5 - \frac{1}{\sqrt{2}}\, \Sigma))\, E + \bar{E}\, (m_2 + (\partial_5 - \frac{1}{\sqrt{2}}\, \Sigma))\, \bar{E}^c) + h.c.\}
\end{aligned} \tag{14.102}$$

where $\bar{E},\ E^c$ have the same electric charge. Choosing the $\mathbb{Z}_2$ parity

$$PE = +E,\ P\bar{E} = +\bar{E},\ PV = +V,\ PE^c = -E^c,\ P\bar{E}^c = -\bar{E}^c,\ P\Sigma = -\Sigma, \tag{14.103}$$

then only states with even parity have massless modes. We thus obtain $N = 1$ SUSY QED in 4D plus an infinite tower of Kaluza-Klein massive modes.

The 5D gravitino $\Psi_M = (\psi_M^1, \psi_M^2)$ decomposes into two 4D gravitini ψ_μ^1, ψ_μ^2 and two dilatini $\psi_5^1,\ \psi_5^2$. To be consistent with the 5D supersymmetry transformations one can assign positive parities to $\psi^1_\mu + \psi^2_\mu$, $\psi^1_5 - \psi^2_5$ and negative parities to $\psi^1_\mu - \psi^2_\mu$, $\psi^1_5 + \psi^2_5$; this assignment partially breaks $\mathcal{N} = 2$ to $\mathcal{N} = 1$ in 4D.

In the next section we shall discuss SUSY $SU(5)$ in 5D on the orbifold $M_4 \times S^1/(\mathbb{Z}_2 \times \mathbb{Z}'_2)$ with the gauge fields and Higgs multiplets in the bulk.

14.8 A Supersymmetric $SU(5)$ Orbifold GUT

Consider the 5D orbifold GUT model of [313, 314].[8] The model has an $SU(5)$ symmetry broken by orbifold parities to the SM gauge group in 4D. The compactification scale $M_c = R^{-1}$ is assumed to be much less than the cutoff scale.

The gauge field is a 5D vector multiplet $\mathcal{V} = (A_M, \lambda_1, \lambda_2, \phi)$, where A_M, ϕ (and their fermionic partners λ_1, λ_2) are in the adjoint representation (**24**) of $SU(5)$. This multiplet consists of one 4D $\mathcal{N} = 1$ supersymmetric vector multiplet $V = (A_\mu, \lambda_1)$ and one 4D chiral multiplet $\Sigma = (\frac{1}{\sqrt{2}}(\phi + iA_5), \lambda_2)$. We also add two 5D hypermultiplets containing the Higgs doublets, $\mathcal{H} = (H_5, H_5{}^c)$, $\overline{\mathcal{H}} = (\bar{H}_{\bar{5}}, \bar{H}^c_{\bar{5}})$.

The orbifold parities for various states in the vector and hyper multiplets are chosen as follows [313, 314] (where we have decomposed all the fields into SM irreducible representations and under $SU(5)$ we have taken $P = (+++++)$, $P' = (---++)$)

States	P	P'	States	P	P'
$V(\mathbf{8}, \mathbf{1}, \mathbf{0})$	$+$	$+$	$\Sigma(\mathbf{8}, \mathbf{1}, \mathbf{0})$	$-$	$-$
$V(\mathbf{1}, \mathbf{3}, \mathbf{0})$	$+$	$+$	$\Sigma(\mathbf{1}, \mathbf{3}, \mathbf{0})$	$-$	$-$
$V(\mathbf{1}, \mathbf{1}, \mathbf{0})$	$+$	$+$	$\Sigma(\mathbf{1}, \mathbf{1}, \mathbf{0})$	$-$	$-$
$V(\bar{\mathbf{3}}, \mathbf{2}, \mathbf{5/3})$	$+$	$-$	$\Sigma(\mathbf{3}, \mathbf{2}, -\mathbf{5/3})$	$-$	$+$
$V(\mathbf{3}, \mathbf{2}, -\mathbf{5/3})$	$+$	$-$	$\Sigma(\bar{\mathbf{3}}, \mathbf{2}, \mathbf{5/3})$	$-$	$+$
$T(\mathbf{3}, \mathbf{1}, -\mathbf{2/3})$	$+$	$-$	$T^c(\bar{\mathbf{3}}, \mathbf{1}, \mathbf{2/3})$	$-$	$+$
$H(\mathbf{1}, \mathbf{2}, +\mathbf{1})$	$+$	$+$	$H^c(\bar{\mathbf{1}}, \mathbf{2}, -\mathbf{1})$	$-$	$-$
$\bar{T}(\bar{\mathbf{3}}, \mathbf{1}, +\mathbf{2/3})$	$+$	$-$	$\bar{T}^c(\mathbf{3}, \mathbf{1}, -\mathbf{2/3})$	$-$	$+$
$\bar{H}(\mathbf{1}, \mathbf{2}, -\mathbf{1})$	$+$	$+$	$\bar{H}^c(\mathbf{1}, \mathbf{2}, +\mathbf{1})$	$-$	$-$

(14.104)

We see the fields supported at the orbifold fixed points $y = 0$ or πR have parities $P = +$ or $P' = +$, respectively. They form complete representations under the $SU(5)$ or SM groups; the corresponding fixed points are called $SU(5)$ and SM "branes." In a 4D effective theory one would integrate out all the massive states, leaving only massless modes of the $P = P' = +$ states. With the above choices of

[8] For other orbifold GUT models in 5D, see [298, 303, 305, 315–318] or 6D, see [319, 320].

orbifold parities, the SM gauge fields and the H and $\bar{H}$ chiral multiplet are the only surviving states in 4D. We thus have an $\mathcal{N}=1$ SUSY in 4D. In addition, the T, $\bar{T}$ and T^c, $\bar{T}^c$ color-triplet states are projected out, solving the doublet-triplet splitting problem that plagues conventional 4D GUTs.

14.9 Gauge Coupling Unification

We follow the field theoretical analysis in [301] (see also [321, 322]).[9] It has been shown there that the threshold correction to a generic gauge coupling due to a tower of KK states with masses $M_{\mathrm{KK}} = m/R$ is given by

$$\alpha^{-1}(\Lambda) = \alpha^{-1}(\mu_0) + \frac{b}{4\pi}\int_{r\Lambda^{-2}}^{r\mu_0^{-2}} \frac{\mathrm{d}t}{t}\,\theta_3\left(\frac{\mathrm{i}t}{\pi R^2}\right), \tag{14.105}$$

where the integration is over the Schwinger parameter, t. The parameters μ_0 and Λ are the IR and UV cut-offs, and $r = \pi/4$ is a numerical factor. θ_3 is the Jacobi theta function, $\theta_3(t) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\pi m^2 t}$, representing the summation over KK states.

For our $S^1/\mathbb{Z}_2$ orbifold there is one modification in the calculation. There are four sets of KK towers, with mass $M_{\mathrm{KK}} = m/R$ (for $P = P' = +$), $(m+1)/R$ (for $P = P' = -$) and $(m+1/2)/R$ (for $P = +, P' = -$ and $P = -, P' = +$), where $m \geq 0$. The summations over KK states give respectively $\frac{1}{2}\left(\theta_3(\mathrm{i}t/\pi R^2) - 1\right)$ for the first two cases and $\frac{1}{2}\theta_2(\mathrm{i}t/\pi R^2)$ for the last two (where $\theta_2(t) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\pi(m+1/2)^2 t}$), and we have separated out the zero modes in the $P = P' = +$ case.

Tracing the renormalization group evolution from low energy scales, we are first in the realm of the MSSM, and the beta function coefficients are $\mathbf{b}^{MSSM} = (-\frac{33}{5}, -1, 3)$. The next energy threshold is the compactification scale M_c. From this scale to the cut-off scale, M_*, we have the four sets of KK states.

Collecting these facts, and using $\theta_2(\mathrm{i}t/\pi R^2) \simeq \theta_3(\mathrm{i}t/\pi R^2) \simeq \sqrt{\frac{\pi}{t}}R$ for $t/R^2 \ll 1$, we find the RG equations,

$$\begin{aligned}\alpha_i^{-1}(M_Z) = \alpha_*^{-1} - \frac{b_i^{MSSM}}{2\pi}\log\frac{M_*}{M_Z} + \frac{1}{4\pi}\left(b_i^{++} + b_i^{--}\right)\log\frac{M_*}{M_c} \\ -\frac{b^{\mathcal{G}}}{2\pi}\left(\frac{M_*}{M_c} - 1\right) + \delta_i^2 + \delta_i^l\end{aligned} \tag{14.106}$$

for $i = 1, 2, 3$, where $\alpha_*^{-1} = \frac{8\pi^2 R}{g_5^2}$ and we have taken the cut-off scales, $\mu_0 = M_c = \frac{1}{R}$ and $\Lambda = M_*$. (Note, this 5D orbifold GUT is a non-renormalizable theory with a cut-off. In string theory, the cut-off will be replaced by the physical

[9] See problem 12.

string scale, M_{STRING}.) The $(++)$ modes include the KK tower of the MSSM gauge sector plus the pair of Higgs doublets, while the $(--)$ modes include the KK tower of the chiral multiplet, Σ, and the charge conjugates Higgs doublets. Hence we find

$$\left(b_i^{++} + b_i^{--}\right) = \frac{2}{3} b_i(V_{MSSM}) + b_i(H,\ \bar{H},\ H^c,\ \bar{H}^c). \tag{14.107}$$

$b^{\mathscr{G}} = \sum_{P=\pm, P'=\pm} b^{\mathscr{G}}_{PP'}$, so in fact it is the beta function coefficient of the orbifold GUT gauge group, $\mathscr{G} = SU(5)$. The beta function coefficients in the last two terms have an $\mathscr{N} = 2$ nature, since the massive KK states enjoy a larger supersymmetry. In general we have $b^{\mathscr{G}} = 2C_2(\mathscr{G}) - 2N_{hyper}T_R$. The first term [in Eq. (14.106)] on the right is the 5D gauge coupling defined at the cut-off scale, the second term accounts for the one loop RG running in the MSSM from the weak scale to the cut-off, the third and fourth terms take into account the KK modes in loops above the compactification scale and the last two terms account for the corrections due to two loop RG running and weak scale threshold corrections.

It should be clear that there is a simple mathematical correspondence to the 4D analysis. We have

$$\begin{aligned} \alpha_G^{-1} \ \ (4D) &\leftrightarrow \alpha_*^{-1} - \frac{b^{\mathscr{G}}}{2\pi}\left(\frac{M_*}{M_c} - 1\right) \ \ (5D) \\ \delta_i^h \ \ (4D) &\leftrightarrow \frac{1}{4\pi}\left(b_i^{++} + b_i^{--}\right)\log\frac{M_*}{M_c} - \frac{b_i^{MSSM}}{2\pi}\log\frac{M_*}{M_G} \ \ (5D). \end{aligned} \tag{14.108}$$

Thus in 5D, the GUT scale threshold corrections determine the ratio M_*/M_c (note the second term in Eq. (14.108) does not contribute to $\delta_s^h = \frac{1}{7}(5\delta_1 - 12\delta_2 + 7\delta_3)$). For $SU(5)$ we have $\mathbf{b}^{++} + \mathbf{b}^{--} = (-6/5, 2, 6)$ and given δ_s^h [Eq. (6.21)] we have

$$\delta_s^h = \frac{12}{28\pi}\log\frac{M_*}{M_c} \approx +0.94 \tag{14.109}$$

or

$$\frac{M_*}{M_c} \approx 10^3. \tag{14.110}$$

If the GUT scale is defined at the point where $\alpha_1 = \alpha_2$, then we have $\delta_1^h = \delta_2^h$ or $\log\frac{M_*}{M_G} \approx 2$. In 5D orbifold GUTs, nothing in particular happens at the 4D GUT scale. However, since the gauge bosons affecting the dimension 6 operators for proton decay obtain their mass at the compactification scale, it is important to realize that the compactification scale is typically lower than the 4D GUT scale and the cut-off is higher (see Fig. 14.5). Let's put in some numbers and see what the scales turn out to be. For example, let's take the color triplet mass $m_T = \frac{1}{2R} = 2 \times 10^{14}$ GeV. We then have the compactification scale, $M_c = 4 \times 10^{14}$ GeV, the cut-off scale,

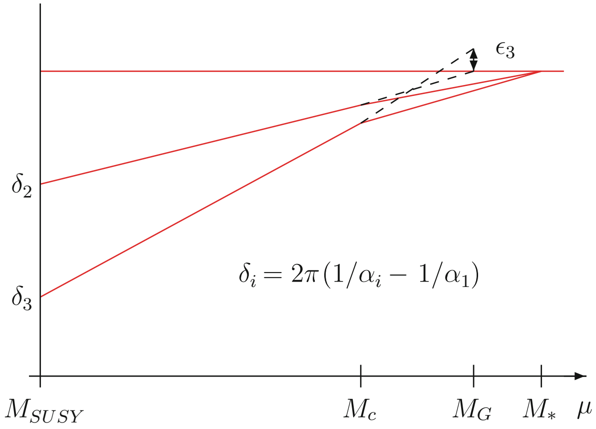

Fig. 14.5 The differences $\delta_i = 2\pi(1/\alpha_i - 1/\alpha_1)$ are plotted as a function of energy scale μ. The threshold correction ϵ_3 defined in the 4D GUT scale is used to fix the threshold correction in the 5D orbifold GUT

$M_* \sim 4 \times 10^{17}$ GeV and the 4D GUT scale, $M_G \sim 5 \times 10^{16}$ GeV. The particular numbers would depend on the threshold corrections at the electroweak scale.

14.10 Quarks and Leptons in 5D Orbifold GUTs

Quarks and lepton fields can be put on either of the orbifold "branes" or in the 5D bulk. If they are placed on the $SU(5)$ "brane" at $y = 0$, then they come in complete $SU(5)$ multiplets. As a consequence couplings of the type given by the first term

$$W \supset \int d^2\theta \int dy\, \delta(y)\, (\bar{H}\; 10\; \bar{5} + H\; 10\; 10) \tag{14.111}$$

will lead to bottom—tau Yukawa unification. This relation is good for the third generation and so it suggests that the third family should reside on the $SU(5)$ brane. Since this relation does not work for the first two families, they might be placed in the bulk or on the SM brane at $y = \pi R$. Without further discussion of quark and lepton masses (see [83, 314, 318, 323] for complete $SU(5)$ or $SO(10)$ orbifold GUT models), let us consider proton decay in orbifold GUTs.

14.11 Proton Decay

Dimension 6 Operators

The interactions contributing to proton decay are those between the so-called X gauge bosons $A_\mu^{(+-)} \in V(+-)$ (where $A_{\mu\, ai}^{(+-)}(x_\mu, y)$ is the five dimensional gauge boson with quantum numbers $(\bar{3}, 2, +5/3)$ under SU(3)× SU(2) × U(1), a and i are color and SU(2) indices respectively) and the $\mathcal{N} = 1$ chiral multiplets on the $SU(5)$ brane at $y = 0$. Assuming all quarks and leptons reside on this brane we obtain the $\Delta B \neq 0$ interactions given by

$$\mathscr{S}_{\Delta B\neq 0} = -\frac{g_5}{\sqrt{2}} \int d^4x A_{\mu\, ai}^{(+-)}(x_\mu, 0) J_{ai}^\mu(x_\mu)^\dagger + h.c. \quad . \tag{14.112}$$

The currents J_{ai}^μ are given by:

$$\begin{aligned} J_{ai}^\mu &= \epsilon_{abc}\, \epsilon_{ij} (u^c)_b^*\, \bar{\sigma}^\mu\, q^{cj} + q_{ai}^*\, \bar{\sigma}^\mu\, e^c - \tilde{l}_i^*\, \bar{\sigma}^\mu\, (d^c)_a \\ &= (u^c)^*\, \bar{\sigma}^\mu\, q + q^*\, \bar{\sigma}^\mu\, e^c - \tilde{l}^*\, \bar{\sigma}^\mu\, d^c \quad , \end{aligned} \tag{14.113}$$

Upon integrating out the X gauge bosons we obtain the effective lagrangian for proton decay

$$\mathscr{L} = -\frac{g_G^2}{2M_X^2} \sum_{i,j} \left[(q_i^* \bar{\sigma}^\mu u_i^c)\, (\tilde{l}_j^* \bar{\sigma}_\mu d_j^c) \; + \; (q_i^* \bar{\sigma}^\mu e_i^c)\, (q_j^* \bar{\sigma}_\mu u_j^c) \right] \quad , \tag{14.114}$$

where all fermions are weak interaction eigenstates and $i, j, k = 1, 2, 3$ are family indices. The dimensionless quantity

$$g_G \equiv g_5 \frac{1}{\sqrt{2\pi R}} \tag{14.115}$$

is the four-dimensional gauge coupling of the gauge bosons zero modes. The combination

$$M_X = \frac{M_c}{\pi} \quad , \tag{14.116}$$

proportional to the compactification scale

$$M_c \equiv \frac{1}{R} \quad , \tag{14.117}$$

is an effective gauge vector boson mass arising from the sum over all the Kaluza-Klein levels:

$$\sum_{n=0}^{\infty} \frac{4}{(2n+1)^2 M_c^2} = \frac{1}{M_X^2} \quad . \tag{14.118}$$

Before one can evaluate the proton decay rate one must first rotate the quark and lepton fields to a mass eigenstate basis. This will bring in both left- and right-handed quark and lepton mixing angles. However, since the compactification scale is typically lower than the 4D GUT scale, it is clear that proton decay via dimension 6 operators is likely to be enhanced.

On the other hand, if quarks and leptons are in the bulk then the dimension 6 operators couple massless quarks to massive KK excitations. In this case proton decay is very much suppressed.

Dimension 5 Operators

The dimension 5 operators for proton decay result from integrating out color triplet Higgs fermions. However in this simplest $SU(5)$ 5D model the color triplet mass is of the form [312]

$$W \supset \int d^2\theta \, dy \, (T(-+)^c \, \partial_y \, T(+-) + \bar{T}(-+)^c \, \partial_y \, \bar{T}(+-)) \tag{14.119}$$

where a sum over massive KK modes is understood. Since only $T, \ \bar{T}$ couple directly to quarks and leptons, *no dimension 5 operators are obtained when integrating out the color triplet Higgs fermions*.

In fact, a $U(1)_R$ symmetry can be defined which prevents dimension 5 proton decay with the $U(1)_R$ charges of the N = 1 superfields given in Table 14.1 [314] and the superpotential has $U(1)_R$ charge, +2. The $U(1)_R$ symmetry is broken when supersymmetry is broken, leaving an unbroken $\mathbb{Z}_2$ subgroup of this $U(1)_R$ which is R parity. Thus dimension 4 baryon and lepton number violating operators are also forbidden.

Table 14.1 $U(1)_R$ charges for 4D vector and chiral superfields

	V	Σ	H	H^c	$\bar{H}$	$\bar{H}^c$	$\mathbf{10}$	$\mathbf{10}^c$	$\bar{\mathbf{5}}$	$\bar{\mathbf{5}}^c$
$U(1)_R$	0	0	0	2	0	2	1	1	1	1

Dimension 4 Baryon and Lepton Violating Operators

If the theory is constructed with an R parity or family reflection symmetry, then no such operators will be generated (see previous discussion).

Summary

In this chapter we have introduced the concept of orbifold GUTs in 5D. We have shown that GUT symmetry breaking and Higgs doublet-triplet splitting can naturally be accomplished via Wilson lines. Thus there is no need for complicated GUT symmetry breaking and doublet-triplet splitting sectors as in 4D GUTs. In addition, one never requires GUT representations larger than the adjoint representation in order to construct "realistic" models. This is nice, since string theories which include gravity, do not contain massless representations of GUT groups larger than the adjoint representation. We have seen that gauge coupling unification is satisfied with GUT threshold corrections coming from the Kaluza-Klein excitations. The 4D GUT scale in these models is not a physical scale. It is replaced instead by the compactification scale. Moreover the compactification scale is typically smaller than the 4D GUT scale. Note, it is the compactification scale which determines the mass of the GUT gauge bosons which mediate proton decay. As a result the dimension 6 operators contributing to proton decay may be enhanced. However the actual proton decay rate depends sensitively on whether the quarks and leptons reside on orbifold fixed points with a local GUT symmetry, in which case the proton decay rate may be enhanced, or the quarks and leptons reside in the bulk, in which case proton decay rates are suppressed (or forbidden entirely [324]). Finally, dimension 5 operators for proton decay can easily be forbidden due to $U(1)$ R symmetries.

Chapter 15
SO(10) SUSY GUT in 5D

We consider a five dimensional supersymmetric SO(10) GUT compactified on an $S^1/(Z_2 \times Z_2')$ orbifold where $S^1 \equiv \mathbb{R}^1/\mathscr{T}$ and $\mathscr{T}$ is the action of translations by $4\pi R$ (see Sect. 14.2). The first orbifold, Z_2, under which $y \to -y$, breaks 5D N=1 supersymmetry (4D N=2) to 4D N=1. The other orbifold, Z_2', under which $y \to -y + \pi R$, breaks SO(10) down to the PS gauge group $\mathrm{SU}(4)_C \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The fundamental domain of the y direction is the line segment $y \in [0, \pi R]$. SO(10) gauge symmetry is present everywhere except at the point $y = \pi R$, which only has Pati-Salam gauge symmetry. Hence, we call the two inequivalent fixed points the "SO(10)" ($y = 0$) and "Pati-Salam" branes ($y = \pi R$) where each fixed point is a three-brane (3+1 dimensional spacetime). The Higgs mechanism on the PS brane completes the breaking of the PS gauge symmetry to the SM gauge group.

The fields which live in the five dimensional space between the branes (known as the "bulk") are even or odd under the orbifold parities. This part of the setup, including the orbifold structure, field parities, and supersymmetry and gauge symmetry breaking are based on work done in [305] and [317]. For more details, please see these references.

We wish to relate some of our fields by a family symmetry, D_3, under which the first and second family fields form a doublet. Other fields within our model will be in various representations of D_3 which will affect the structure of the Yukawa matrices we generate. We take the family symmetry to be independent of the orbifold symmetries. A brief summary of the D_3 group is provided in Sect. 10.4, where we give the information necessary to understand the family representations and couplings used in our model.

The 5D supersymmetric vector multiplet $\mathscr{V} = (A_M, \lambda_1, \lambda_2, \phi)$ contains a 4D vector multiplet $V = (A_\mu, \lambda_1)$ and a 4D chiral adjoint $\Sigma = ((\phi + iA_5)/\sqrt{2}, \lambda_2)$ [see Eq. (14.97)]. For a generic hypermultiplet $\mathscr{H} = (h, h^c, \tilde{h}, \tilde{h}^c)$ which breaks up into the 4D chiral multiplets $H = (h, \tilde{h})$ and $H^c = (h^c, \tilde{h}^c)$ [see Eq. (14.92)], we have the 5D action [see Eq. (14.99)].

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We now briefly summarize the gauge unification results given by Kim and Raby [317]. A 5D gauge theory is non-renormalizable and gets large corrections at the cutoff. Corrections to gauge couplings, however, will be the same for all couplings unified into the larger gauge group. These corrections will affect the absolute values of the gauge couplings, but not the differences. Further, if the gauge symmetry is broken only by orbifolding or by a Higgs mechanism on the branes, the differences in the couplings will have a logarithmic, calculable running.

The states which affect the differential running are the bulk vector multiplet $\mathscr{V}$ and the bulk Higgs hypermultiplet $\mathscr{H}$.[1] The placement and number of complete matter multiplets does not affect gauge coupling unification, since matter multiplets (16s of SO(10)) act equally across the three gauge couplings and cannot affect the coupling differences. Those states in the theory outside of the MSSM have twisted orbifold boundary conditions and so have masses at the compactification scale (M_c). For energies below M_c, the theory is the MSSM. The effects of running between M_c and M_* (the cutoff scale), including the Kaluza-Klein (KK) towers, are taken as threshold corrections at M_c. Without these threshold corrections, it is known that the MSSM unifies around $M_G \sim 3 \times 10^{16}$ GeV with a coupling of $\alpha_{\mathrm{GUT}} \sim 1/24$ and a GUT-scale threshold correction for α_3 of $\varepsilon_3 \sim -0.04$. Assuming unification in the orbifold theory at the cutoff scale M_* and that the PS breaking Higgs VEV is of order the cutoff scale, we can solve for M_* and M_c in terms of the 4D GUT parameters M_G, α_{GUT}, and ε_3. This leads to $M_* \sim 10^{17}$ GeV and $M_c \sim 10^{14}$ GeV [317].

The matter field locations are constrained by proton decay:

- Matter fields on the SO(10) brane
  There are gauge bosons within SO(10) which mediate baryon (B) and lepton (L) number-violating interactions. All of these are outside of the Standard Model and hence have masses of order M_c or higher. After integrating out these fields (and their KK modes), we get dimension six operators which violate B and L for any matter multiplets on the SO(10) brane. These operators are suppressed by $1/M_c^2$. Given $M_c \sim 10^{14}$ GeV, current bounds on proton decay rule out models which have these operators for the first and second families. Thus only the third family can reside on the SO(10) brane [103].[2] Dimension 5 proton decay operators

[1] When considering differential running of the gauge couplings, a Higgs hypermultiplet in the bulk is effectively the same as a 4D 10 of SO(10) with light Higgs MSSM doublets and heavy Higgs triplets of mass M_c. This setup admits gauge coupling unification as shown by Kim and Raby [317]. In particular, see the calculations leading to Eq. (3.13) of that paper. Effects from brane Higgs doublets would be felt up to M_* and would tend to inhibit unification since they drive the couplings apart rather than together.

[2] With dimension six operators for the third family, mixing between this family and the first two can induce proton decay. Assuming that the mixing is of order $|V_{ub}|$ or $|V_{cb}|$ and that the gauge bosons have mass at M_c, naive calculations using formulae in [103] put the proton lifetime many orders of magnitude above current limits, since this leads to an effective gauge boson mass of order $M_c/(V_{cb}V_{ub}) \approx 6 \times 10^{17}$ GeV.

vanish since the color triplet Higgs states obtain off-diagonal mass with triplets in $\overline{10}$ and these states do not couple to matter.

- Matter fields on the PS brane or in the bulk
Pati-Salam gauge symmetry does not relate the left-handed fields ψ ((4, 2, 1) in PS) to the right-handed fields ψ^c (($\bar{4}$, 1, 2) in PS), and we do not get baryon number-violating dimension six operators after integrating out the heavy gauge bosons. Therefore, any matter fields can be on the PS brane as long as the PS breaking scale is not extremely low. This scale in our theory is $M_* \sim 10^{17}$ GeV, and proton decay is not a problem here.
In principle, we can consider higher dimensional operators with derivative interactions $\partial_5 = \partial/\partial y$. Because the coefficients of the higher dimensional operators are not determined from the theory we cannot calculate the proton decay rate from these operators accurately. However, we can get a bound that is consistent with our setup by assuming unknown coefficients to be order one. See Kim and Raby for more details [317].

Proton decay constrains the first and second families to reside either in the bulk or on the PS brane, but does not constrain the location of the third family. We choose to place all three families on the PS brane.

Let us summarize the basic setup.

- Gauge symmetry : SO(10) in the bulk and at $y = 0$, Pati-Salam at $y = \pi R$.
- Higgs fields come from 10 dimensional hypermultiplets in the bulk.
- First and second family matter fields, a doublet under D_3, are on the PS brane. The third family is a singlet under D_3 and also sits on the PS brane.

15.1 Yukawa Matrices

This section introduces the fields and superpotential of our model. Here we calculate the Yukawa matrices associated with the massless fields corresponding to the Standard Model fermions. This section is based on the work in the Appendix of Kim et al. [318].

We present here a 5D version of a 4D model by Dermisek and Raby in [146] which itself was based on prior works [96, 139, 140, 216, 325]. Our purpose is to illustrate how such a 4D model could be placed into a 5D context and what advantages can be gained from the use of the extra dimension.

In using the extra dimension, we have separated the fields on the PS brane from the important mass parameter $M_\chi = M_0(1 + \alpha X)$ which gives much of the distinction between the particle types. There are matter fields on the PS brane, and bulk matter fields which mediate between these brane fields and the SO(10) breaking M_χ VEV in such a way to give us the desired Yukawa matrix elements.

The states are placed as follows. In the bulk, we have eight matter hypermultiplets, forming four doublets under D_3 and transforming as 16s under SO(10). Each of these hypermultiplet doublets has a different parity under the orbifold. The Higgs,

Table 15.1 Bulk fields

Field	PS Symm	D_3 Symm
$16 = \begin{pmatrix} \psi_{++} \\ \psi^c_{+-} \end{pmatrix}$	$\begin{pmatrix} (4,2,1) \\ (\bar{4},1,2) \end{pmatrix}$	$\mathbf{2_A}$
$\overline{16} = \begin{pmatrix} \overline{\psi}_{--} \\ \overline{\psi^c}_{-+} \end{pmatrix}$	$\begin{pmatrix} (\bar{4},2,1) \\ (4,1,2) \end{pmatrix}$	$\mathbf{2_A}$
$16' = \begin{pmatrix} \psi'_{+-} \\ \psi^{c\prime}_{++} \end{pmatrix}$	$\begin{pmatrix} (4,2,1) \\ (\bar{4},1,2) \end{pmatrix}$	$\mathbf{2_A}$
$\overline{16'} = \begin{pmatrix} \overline{\psi'}_{-+} \\ \overline{\psi^{c\prime}}_{--} \end{pmatrix}$	$\begin{pmatrix} (\bar{4},2,1) \\ (4,1,2) \end{pmatrix}$	$\mathbf{2_A}$
$\widetilde{16} = \begin{pmatrix} \widetilde{\psi}_{-+} \\ \widetilde{\psi^c}_{--} \end{pmatrix}$	$\begin{pmatrix} (4,2,1) \\ (\bar{4},1,2) \end{pmatrix}$	$\mathbf{2_A}$
$\widetilde{\overline{16}} = \begin{pmatrix} \widetilde{\overline{\psi}}_{+-} \\ \widetilde{\overline{\psi^c}}_{++} \end{pmatrix}$	$\begin{pmatrix} (\bar{4},2,1) \\ (4,1,2) \end{pmatrix}$	$\mathbf{2_A}$
$\widetilde{16'} = \begin{pmatrix} \widetilde{\psi'}_{--} \\ \widetilde{\psi^{c\prime}}_{-+} \end{pmatrix}$	$\begin{pmatrix} (4,2,1) \\ (\bar{4},1,2) \end{pmatrix}$	$\mathbf{2_A}$
$\widetilde{\overline{16'}} = \begin{pmatrix} \widetilde{\overline{\psi'}}_{++} \\ \widetilde{\overline{\psi^{c\prime}}}_{+-} \end{pmatrix}$	$\begin{pmatrix} (\bar{4},2,1) \\ (4,1,2) \end{pmatrix}$	$\mathbf{2_A}$
$10 = \begin{pmatrix} H_{++} \\ H^c_{+-} \end{pmatrix}$	$\begin{pmatrix} (1,2,2) \\ (6,1,1) \end{pmatrix}$	$\mathbf{1_A}$
$\overline{10} = \begin{pmatrix} \overline{H}_{--} \\ \overline{H^c}_{-+} \end{pmatrix}$	$\begin{pmatrix} (1,2,2) \\ (6,1,1) \end{pmatrix}$	$\mathbf{1_A}$

as before, is contained inside a 10 hypermultiplet of SO(10). These fields are listed in Table 15.1. On the SO(10) brane there is only the mass parameter M_χ, which is a singlet under D_3 and is a mix of singlet and X under SO(10): $M_\chi = M_0(1+\alpha X)$.[3] On the Pati-Salam brane, we have three sets of left- and right-handed matter fields. Two of these sets form a doublet under D_3. In addition, there are several extra fields: ϕ_a, $\widetilde{\phi}_a$, A, A_{15}, Φ_L, and Φ_R. Of these extra fields, those with subscript a are D_3 doublets. The rest are $\mathbf{1_A}$ save A which is $\mathbf{1_B}$ under D_3. All of these extra fields get nonzero VEVs except for $\langle\widetilde{\phi}_1\rangle = 0$. All extra fields are SO(10) singlets, save A_{15} which is an SU(4) adjoint and gets a VEV $\langle A_{15}\rangle = \langle A^0_{15}\rangle\,(B-L)$. The PS brane fields are listed in Table 15.2.

[3]The $SO(10)$ breaking mass term, M_χ, can be due to an additional 16, $\overline{16}$ which obtain VEVs in the right-handed neutrino direction and couple, for example, as follows, $\widetilde{\overline{16}}_a\langle 16\,\overline{16}\rangle 16'_a$.

Table 15.2 PS Brane fields

Field	*PS* Symm	D_3 Symm
ψ	$(4, 2, 1)$	$\mathbf{2_A}$
ψ^c	$(\bar{4}, 1, 2)$	$\mathbf{2_A}$
ψ_3	$(4, 2, 1)$	$\mathbf{1_A}$
ψ_3^c	$(\bar{4}, 1, 2)$	$\mathbf{1_A}$
ϕ	$(1, 1, 1)$	$\mathbf{2_A}$
$\widetilde{\phi}$	$(1, 1, 1)$	$\mathbf{2_A}$
A	$(1, 1, 1)$	$\mathbf{1_B}$
A_{15}	$(15, 1, 1)$	$\mathbf{1_A}$
Φ_L	$(1, 1, 1)$	$\mathbf{1_A}$
Φ_R	$(1, 1, 1)$	$\mathbf{1_A}$

We choose the following superpotential:

$$
\begin{aligned}
W_1 &= \delta(y-\pi R)\left\{\lambda_1 \psi_3 H \psi_3^c + \lambda_2 \psi_a H \left(\psi^{c\prime}_{++}\right)_a \Phi_R + \lambda_2 \left(\psi_{++}\right)_a H \psi_a^c \Phi_L\right\} \\
W_2 &= \overline{16}_a \partial_y 16_a + \overline{16'}_a \partial_y 16'_a + \overline{\widetilde{16}}_a \partial_y \widetilde{16}_a + \overline{\widetilde{16'}}_a \partial_y \widetilde{16'}_a + \overline{10} \partial_y 10 \\
&\quad + \delta(y)\left\{\overline{\widetilde{16}}_a M_\chi 16'_a + \overline{\widetilde{16'}}_a M_\chi 16_a\right\} \\
&\quad + \delta(y-\pi R)\left\{\left(\overline{\widetilde{\psi'}}_{++}\right)_a \left[\lambda_3 A_{15} \phi_a \psi_3 + \lambda_4 A_{15} \widetilde{\phi}_a \psi_a + \lambda_5 A \psi_a\right] \Phi_L \right. \\
&\quad \left. + \left(\overline{\widetilde{\psi^c}}_{++}\right)_a \left[\lambda_3 A_{15} \phi_a \psi_3^c + \lambda_4 A_{15} \widetilde{\phi}_a \psi_a^c + \lambda_5 A \psi_a^c\right] \Phi_R\right\}
\end{aligned} \tag{15.1}
$$

There are 12 U(1) symmetries associated with our superpotential. We choose to parametrize these by allowing the following 12 fields to be charged each under different U(1)s: ψ_3, ψ_3^c, H^c_{+-}, M_χ, $\left(\widetilde{\psi}'_{--}\right)_a$, $\left(\widetilde{\psi}^{c\prime}_{-+}\right)_a$, $\left(\widetilde{\psi}_{-+}\right)_a$, Φ_L, A_{15}. After specifying the U(1) symmetries of the above fields, the U(1) transformations of all other fields are uniquely defined. There are in addition 2 Z_2 symmetries. First is the Z_2 orbifold parity under $y \to -y$. The transformations of the bulk fields under this symmetry have already been defined in Table 15.1. In addition, let the rest of the independent fields: ψ_3, ψ_3^c, M_χ, Φ_L, A_{15} be even under this symmetry. There is also a Wilson line which breaks $SO(10)$ to PS as defined in Sect. 14.4. The second Z_2 involves a sign ambiguity in the transformation of Φ_R. Under a given symmetry, if $\Phi_L \to e^{i\alpha}\Phi_L$, then the superpotential terms imply that $\Phi_R \to e^{in\pi}e^{i\alpha}\Phi_R$ with $n = 0, 1$. We choose to require that $n = 1$ here in order to forbid terms created by the replacement of Φ_L by Φ_R or vice-versa. We also assume that $\langle \Phi_L \rangle / M_* \ll 1$ so that replacements like $\Phi_L \to \Phi_R^2$ are negligible.[4] Let the 12 independent fields be

[4]Such terms given by the substitution $\Phi_L \to \Phi_R^2$ or $\Phi_R \to \Phi_L^2$ would lead to a Yukawa matrix structure different from the one desired and so should be forbidden by some symmetry.

uncharged under this symmetry. We require all of these symmetries just listed to be symmetries of the theory so as to forbid unwanted extra terms in the superpotential.

The superpotential has a left-right symmetry, under which $\psi_3 \leftrightarrow \psi_3^c$, $\psi \leftrightarrow \psi^c$, $\begin{pmatrix} 16 \\ \overline{16} \end{pmatrix} \leftrightarrow \begin{pmatrix} 16' \\ \overline{16'} \end{pmatrix}$, $\begin{pmatrix} \widetilde{16} \\ \widetilde{\overline{16}} \end{pmatrix} \leftrightarrow \begin{pmatrix} \widetilde{16'} \\ \widetilde{\overline{16'}} \end{pmatrix}$, $\Phi_L \leftrightarrow \Phi_R$. This symmetry, which commutes with the family D_3 symmetry, is broken spontaneously by the VEVs of Φ_L and Φ_R, which we require to be slightly different. This difference is encapsulated by a small parameter η:

$$\langle \Phi_R \rangle = \langle \Phi_L \rangle (1 + \eta). \tag{15.2}$$

We now turn to the equations of motion for the left-handed states:

$$\begin{aligned}
&\frac{\partial W}{\partial \left(\overline{\psi}_{--}\right)_a} = 0 \Longrightarrow \partial_y \left(\psi_{++}\right)_a = 0 \\
&\frac{\partial W}{\partial \left(\overline{\psi'}_{-+}\right)_a} = 0 \Longrightarrow \partial_y \left(\psi'_{+-}\right)_a = 0 \\
&\frac{\partial W}{\partial \left(\widetilde{\overline{\psi}}_{+-}\right)_a} = 0 \Longrightarrow \partial_y \left(\widetilde{\psi}_{-+}\right)_a + \delta(y) M_\chi \left(\psi'_{+-}\right)_a = 0 \\
&\frac{\partial W}{\partial \left(\widetilde{\overline{\psi}'}_{++}\right)_1} = 0 \Longrightarrow \partial_y \left(\widetilde{\psi}'_{--}\right)_2 + \delta(y) M_\chi \left(\psi_{++}\right)_2 \\
&\qquad\qquad + \delta(y - \pi R) \left[\lambda_3 A_{15} \phi_2 \psi_3 + \lambda_4 A_{15} \widetilde{\phi}_1 \psi_1 + \lambda_5 A \psi_2\right] \Phi_L = 0 \\
&\frac{\partial W}{\partial \left(\widetilde{\overline{\psi}'}_{++}\right)_2} = 0 \Longrightarrow \partial_y \left(\widetilde{\psi}'_{--}\right)_1 + \delta(y) M_\chi \left(\psi_{++}\right)_1 \\
&\qquad\qquad + \delta(y - \pi R) \left[\lambda_3 A_{15} \phi_1 \psi_3 + \lambda_4 A_{15} \widetilde{\phi}_2 \psi_2 - \lambda_5 A \psi_1\right] \Phi_L = 0
\end{aligned} \tag{15.3}$$

Solving these equations leads to knowledge of the overlap between the massless fields and the original fields in the Lagrangian. The following relationships only have the massless components on the right hand sides of the equations. (We have replaced the brane fields by their VEVs)

$$\begin{aligned}
&\left(\psi'_{+-}\right)_a \supset 0 \\
&\left(\widetilde{\psi}_{-+}\right)_a \supset 0 \\
&\left(\widetilde{\psi}'_{--}\right)_1 \supset \frac{1}{2} \varepsilon_{--}(y) \left[\lambda_3 \langle A_{15} \rangle \langle \phi_1 \rangle \psi_3 + \lambda_4 \langle A_{15} \rangle \langle \widetilde{\phi}_2 \rangle \psi_2 - \lambda_5 \langle A \rangle \psi_1\right] \langle \Phi_L \rangle \\
&\left(\widetilde{\psi}'_{--}\right)_2 \supset \frac{1}{2} \varepsilon_{--}(y) \left[\lambda_3 \langle A_{15} \rangle \langle \phi_2 \rangle \psi_3 + \lambda_4 \langle A_{15} \rangle \langle \widetilde{\phi}_1 \rangle \psi_1 + \lambda_5 \langle A \rangle \psi_2\right] \langle \Phi_L \rangle
\end{aligned} \tag{15.4}$$

$$(\psi_{++})_1 \supset -\frac{1}{M_\chi}\left[\lambda_3 \langle A_{15}\rangle \langle\phi_1\rangle \psi_3 + \lambda_4 \langle A_{15}\rangle \left\langle\widetilde{\phi_2}\right\rangle \psi_2 - \lambda_5 \langle A\rangle \psi_1\right] \langle\Phi_L\rangle$$

$$(\psi_{++})_2 \supset -\frac{1}{M_\chi}\left[\lambda_3 \langle A_{15}\rangle \langle\phi_2\rangle \psi_3 + \lambda_4 \langle A_{15}\rangle \langle\widetilde{\phi_1}\rangle \psi_1 + \lambda_5 \langle A\rangle \psi_2\right] \langle\Phi_L\rangle$$

where

$$\varepsilon_{--}(y) \equiv \begin{cases} +1 & \text{for } y \in [-2\pi R, -\pi R] \\ -1 & \text{for } y \in [-\pi R, 0] \\ +1 & \text{for } y \in [0, \pi R] \\ -1 & \text{for } y \in [\pi R, 2\pi R] \end{cases}$$

The equations for the right-handed states can be obtained from the left-right symmetry present in the model. We list the most important of these equations:

$$\left(\psi^{c\prime}_{++}\right)_1 \supset -\frac{1}{M_\chi}\left[\lambda_3 \langle A_{15}\rangle \langle\phi_1\rangle \psi^c_3 + \lambda_4 \langle A_{15}\rangle \left\langle\widetilde{\phi_2}\right\rangle \psi^c_2 - \lambda_5 \langle A\rangle \psi^c_1\right] \langle\Phi_R\rangle$$

$$\left(\psi^{c\prime}_{++}\right)_2 \supset -\frac{1}{M_\chi}\left[\lambda_3 \langle A_{15}\rangle \langle\phi_2\rangle \psi^c_3 + \lambda_4 \langle A_{15}\rangle \langle\widetilde{\phi_1}\rangle \psi^c_1 + \lambda_5 \langle A\rangle \psi^c_2\right] \langle\Phi_R\rangle \tag{15.5}$$

The three ψ_i fields span the space of the left-handed massless states. Because the other fields which we've integrated out have massless components, the kinetic energy and gauge interaction terms for the ψ_i fields are no longer orthonormal. The same is true for the ψ_i^c states by the left-right symmetry. We will discuss the effects of rotating and rescaling these fields to an orthonormal basis later.

We replace the $(\psi_{++})_a$ and $\left(\psi^{c\prime}_{++}\right)_a$ fields in W_1 by their corresponding massless parts in order to get the low energy Yukawa matrices. The result[5]:

$$Y_u = \begin{pmatrix} 0 & \varepsilon'\rho & \varepsilon\xi \\ -\varepsilon'\rho & \tilde{\varepsilon}\rho & \varepsilon \\ \varepsilon\xi & \varepsilon & 1 \end{pmatrix} \lambda \tag{15.6}$$

$$Y_d = \begin{pmatrix} 0 & \varepsilon' & \varepsilon\sigma\xi \\ -\varepsilon' & \tilde{\varepsilon} & \varepsilon\sigma \\ \varepsilon\xi & \varepsilon & 1 \end{pmatrix} \lambda$$

$$Y_e = \begin{pmatrix} 0 & -\varepsilon' & 3\varepsilon\xi \\ \varepsilon' & 3\tilde{\varepsilon} & 3\varepsilon \\ 3\varepsilon\sigma\xi & 3\varepsilon\sigma & 1 \end{pmatrix} \lambda$$

[5] We have chosen to use the notation found in [146] to ease comparison between prior works and our own.

Definitions follow for these variables, where we have used $M_\chi = M_0(1 + \alpha X)$, $\langle \Phi_R \rangle = \langle \Phi_L \rangle (1 + \eta)$, and have added factors of the cutoff scale in order to make the couplings λ_i all dimensionless. We have also assumed $\eta \ll 1$ and $\alpha \sim \mathscr{O}(1)$.

$$\begin{aligned}
\tilde{\varepsilon} &\equiv \frac{\lambda_2 \lambda_4 \left\langle A^0_{15} \right\rangle \left\langle \widetilde{\phi_2} \right\rangle \langle \Phi_L \rangle^2}{3\lambda_1 M_*^3 M_0} \frac{4\alpha}{(1+\alpha)(1-3\alpha)} \\
\varepsilon' &\equiv -\frac{\lambda_2 \lambda_5 \langle A \rangle \langle \Phi_L \rangle^2}{\lambda_1 M_*^2 M_0} \frac{4\alpha}{(1+\alpha)(1-3\alpha)} \\
\rho &\equiv \frac{2\eta(1-3\alpha)}{4\alpha} \\
\varepsilon &\equiv -9\lambda_3 \langle \phi_1 \rangle \frac{\lambda_2 \left\langle A^0_{15} \right\rangle \langle \Phi_L \rangle^2}{3\lambda_1 M_*^3 M_0} \frac{1}{(1+\alpha)} \\
\xi &\equiv \frac{\langle \phi_2 \rangle}{\langle \phi_1 \rangle} \\
\sigma &\equiv \frac{1+\alpha}{1-3\alpha} \\
\lambda &\equiv \lambda_1 \sqrt{\frac{2M_c}{\pi M_*}}
\end{aligned} \tag{15.7}$$

These Yukawa matrices are the same as those found in [216], except for the (1,3) and (3,1) elements. It has been shown in [136] that (1,3) elements are needed in models of this kind in order to fit $\sin 2\beta$. The Yukawa matrices are identical to those in [146]. Note, the parameter $\rho \propto \eta$ or the small amount of left-right symmetry breaking. It is this small parameter which allows $m_u < m_d$ even though $m_t \gg m_b$. In Fig. 15.1 we list the lowest-order diagrams which give the Yukawa matrices. Each diagram is followed by the element(s) to which it contributes. In our analysis we have used a basis in which the massless matter fields are not orthonormal. Rotation and rescaling to a canonical orthonormal basis would in general introduce changes to the Yukawa matrices. We have neglected effects from this change of basis, and our justification for this is the following. Were a fit to be done with these effects included, the input Yukawa parameters would compensate by changing their values. We assume that the input parameters could compensate to the extent that we would obtain essentially the same fit in this case as in the case we have presented here without these extra effects. We leave it to further research to explore whether this assumption is a good one.

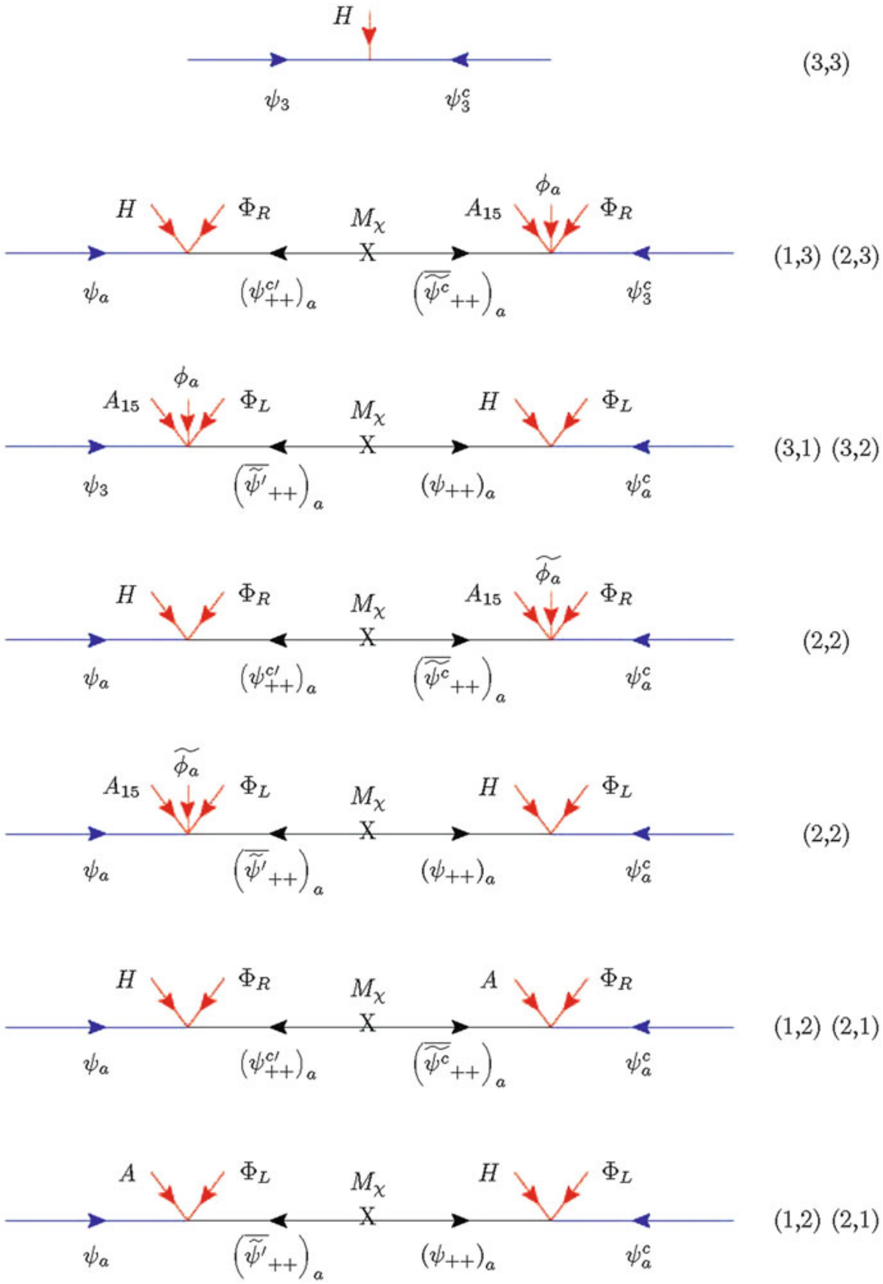

Fig. 15.1 Effective Yukawa mass operators. *This figure is reproduced from the appendix of [318]*

Summary

In this chapter we have constructed a 5D $SO(10)$ orbifold SUSY GUT which reproduces the Yukawa matrices discussed earlier in Sect. 10.3. $SO(10)$ symmetry breaking and Higgs doublet-triplet splitting is accomplished via a Wilson line which breaks $SO(10)$ to the Pati-Salam gauge symmetry. Then PS is broken further to the SM gauge group via a Higgs VEV on the PS brane, [317]. The three families of quarks and leptons were placed on the PS brane. Hence proton decay due to dimension 6 operators is suppressed. In addition, dimension 5 baryon and lepton number violating operators are not generated via color triplet Higgs exchange. Finally the theory preserves an R parity. Thus proton decay rates are severely suppressed in this model.

Gauge coupling unification is satisfied with threshold corrections due to K-K modes at the compactification scale. Moreover, Yukawa unification with a D_3 family symmetry allows for a very predictive theory. Finally, the small ρ parameter, Eq. (15.7), which is necessary to understand why $m_u/m_d < 1$, while $m_t/m_b \gg 1$ is due to the left-right symmetry breaking VEV of $\Phi_{L,R}$, Eq. (15.2).

Chapter 16
An E_6 Orbifold GUT in 5D

In this section we construct a novel 5D orbifold GUT with an E_6 gauge symmetry. We take the 5D gauge field, given by (V, Σ), in the adjoint representation $(\mathbf{78})$ of E_6. In addition to this we add a matter hypermultiplet $(\mathbf{27} + \overline{\mathbf{27}})$.

We define two orbifold parities

$$P = \exp(\pi \mathrm{i} Q_Z/3) \times P_F\,, \qquad P' = \exp[3\pi \mathrm{i}(B-L)/2] \times P'_F\,, \tag{16.1}$$

which break the E_6 via P to SO_{10} and then via P' to PS.[1] Q_Z is the abelian charge in E_6 commuting with SO_{10}, normalized such that the $\mathbf{27}$ decomposes to $\mathbf{16}_1 + \mathbf{10}_{-2} + \mathbf{1}_4$, and P_F, P'_F are appropriate discrete flavor charges. (For explicit definition of the parities in the corresponding string model, see Sect. 20.4.) It is easy to obtain the following projections to $(++)$ modes, where the first step follows from P alone and the second follows from the subsequent action of P',

$$\begin{aligned} V = \mathbf{78} \to &\quad \mathbf{45} \to \quad \text{adjoint of PS}\,, \\ \Sigma = \mathbf{78} \to &\ \mathbf{16} + \overline{\mathbf{16}} \to \bar{\mathscr{Q}}_3 + \overline{\chi}^c\,, \\ \mathbf{27} \to &\quad \mathbf{16} \to \quad \mathscr{Q}_3\,, \\ \overline{\mathbf{27}} \to &\quad \mathbf{10} \to \quad \mathscr{H}\,. \end{aligned} \tag{16.2}$$

In this equation, we have identified the third family of quarks and leptons as well as the MSSM Higgs-doublet pair ($\mathscr{H} = H_u + H_d$ where H_u and H_d are the MSSM Higgs doublets responsible for the up- and down-type quark/charged lepton masses),

$$\bar{\mathscr{Q}}_3 = (\bar{\mathbf{4}}, \mathbf{1}, \bar{\mathbf{2}}), \qquad \mathscr{Q}_3 = (\mathbf{4}, \mathbf{2}, \mathbf{1}), \qquad \mathscr{H} = (\mathbf{1}, \mathbf{2}, \bar{\mathbf{2}}). \tag{16.3}$$

[1] In this breaking scheme, the brane at $y = 0$ has an $SO(10)$ symmetry, whereas the brane at $y = \pi R$ has an $SU(6) \times SU(2)_R$ symmetry. The overlap is Pati–Salam. See Fig. 16.1.

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As a consequence of the fact that the third family and Higgs doublet come from the bulk gauge and **27** hypermultiplets we obtain a gauge-Yukawa unification relation,

$$\lambda_t = \lambda_b = \lambda_\tau = \lambda_{\nu_\tau} = g_{4\mathrm{D}} \equiv \sqrt{4\pi\alpha_{\mathrm{GUT}}}\,, \tag{16.4}$$

where $g_{4\mathrm{D}}$ is the 4D gauge coupling constant at the compactification scale. This relation can be seen by inspecting the 5D bulk gauge interaction

$$\int_0^{\pi R} dx^5 \left(\overline{\mathbf{27}}\ \Sigma\ \mathbf{27}\right) \to g_{4\mathrm{D}}\, \bar{\mathscr{Q}}_3\ \mathscr{H}\ \mathscr{Q}_3\,, \tag{16.5}$$

where $g_{4\mathrm{D}} = \frac{g_{5\mathrm{D}}}{\sqrt{\pi R}}$.

Of course, we then need to spontaneously break PS to the SM via the standard Higgs mechanism. This can be accomplished when the "right-handed neutrino" fields in

$$\bar{\chi} = (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}), \qquad \overline{\chi}^c = (\mathbf{4}, \mathbf{1}, \mathbf{2}) \tag{16.6}$$

obtain non-vanishing vacuum expectation values (vevs)

$$\langle \bar{\nu} \rangle_{\bar{\chi}} = \langle \overline{\nu}^c \rangle_{\overline{\chi}^c} = M_{\mathrm{PS}}. \tag{16.7}$$

We already have one such state but we need more (if only for anomaly cancellation). Consider the addition of three more **27** hypermultiplets given by $3 \times (\mathbf{27} + \overline{\mathbf{27}})$. Upon applying the orbifold parities we find

$$3 \times (\mathbf{27} + \overline{\mathbf{27}}) \to 2(\mathbf{16}) + \overline{\mathbf{16}} + 3(\mathbf{10}) \to 2(\bar{\chi}) + \overline{\chi}^c + 3(C), \tag{16.8}$$

where $C = (\mathbf{6}, \mathbf{1}, \mathbf{1})$. We now have a total of $2(\bar{\chi} + \overline{\chi}^c)$ fields. Note, with one C, one $\bar{\chi},\ \bar{\chi}^c$ pair and a superpotential[2] given by

$$\mathscr{W} = \bar{\chi}\bar{\chi}C + \overline{\chi}^c\overline{\chi}^c C, \tag{16.9}$$

we can give mass to the color triplets and also break PS to the SM along a D-and F-flat direction. (The D-flatness condition requires $\langle \bar{\nu} \rangle_{\bar{\chi}} = \langle \overline{\nu}^c \rangle_{\overline{\chi}^c}$.) In the end, however, we must guarantee that the extra $\bar{\chi},\ \bar{\chi}^c,\ C$ states obtain mass above the PS breaking scale.

But what about the first two families? When constructing an orbifold GUT, one has the option of placing the first two families in the bulk or on either brane. One of the main considerations is to avoid rapid proton decay due to gauge exchange and another is to generate a hierarchy of fermion masses. If the compactification

[2] This superpotential could, in principle, be localized on the $SO(10)$ brane.

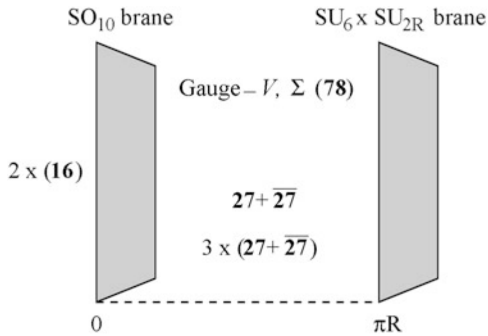

Fig. 16.1 5D E_6 orbifold GUT model with bulk and brane states. The bulk gauge symmetry is broken to SO_{10} on the end of world brane at $x^5 = 0$ and to $SU_6 \times SU_{2R}$ at $x^5 = \pi R$. The massless sector of the 4D effective theory has a PS gauge symmetry. In addition, the bulk contains four hypermultiplets, and the SO_{10} brane contains two spinor representations, giving rise to the first two matter families. *Reprinted from Nuclear Physics B 704, T. Kobayashi, S. Raby, R.-J. Zhang, "Searching for realistic 4d string models with a Pati–Salam symmetry. Orbifold grand unified theories from heterotic string compactification on a Z6 orbifold," Page 10, Copyright (2005), with permission from Elsevier*

scale is much smaller than the GUT scale, say $M_c \ll M_{\mathrm{GUT}}$, then it is not possible to place the first two families on the SO_{10} brane. It would however be fine to place them in the bulk or on the $SU_6 \times SU_{2R}$ brane, since in the first case the families are in irreducible representations with massive KK modes, while in the latter case one family is contained in two irreducible representations $(\mathbf{15}, \mathbf{1}) + (\bar{\mathbf{6}}, \mathbf{2})$.[3] In both cases, gauge exchange takes massless quarks and leptons into massive states. Hence there is *no* problem with proton decay. If however $M_c \geq M_{\mathrm{GUT}}$ then one can place the first two families on either brane. Unfortunately, in string theory, we do not get to choose *easily* where to place the families. It is determined by the choice of vacuum. In the heterotic string version of the model we find two families sitting on the SO_{10} brane, as in Fig. 16.1.

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[3] In the latter case we would need to give mass to the extra states which are not in the MSSM.

Chapter 17
SUSY Breaking in 5D

Scherk-Schwarz Symmetry Breaking

The 5D N = 1 SUSY theory, equivalent to N = 2 in 4D, has a non-Abelian $SU(2)_R$ symmetry in which the two gauginos, (λ_1, λ_2), in the N = 2 vector supermultiplet and the two scalars, $(h, h^{c\dagger})$, in the N = 2 hypermultiplet transform as doublets, while the other states transform as singlets. It was shown that SUSY can be broken via orbifolding and using the holonomy associated with a $U(1)_R$ subgroup of $SU(2)_R$ [326, 327]. In early versions of this mechanism, the SUSY breaking scale was linked to the compactification scale which is not consistent with SUSY as a solution to the gauge hierarchy problem. However, recently a variation of the Scherk-Schwarz mechanism has been shown to allow for a SUSY breaking scale much below the compactification scale [314, 328]. In addition, it has also been shown in [329, 330] that this mechanism is equivalent to radion F-term SUSY breaking or by $SU(2)_R$ Wilson line breaking in [331]. For a nice review on symmetry breaking in extra dimensions, see [332].

Following [328], consider a simple example on $M_4 \times S_1/\mathbb{Z}_2$ with the gauge and Higgs multiplets in the 5D bulk. We have the N = 1 4D superfields, $\{V, \Sigma\}$ in the adjoint representation of $SU(5)$ for the 5D gauge multiplet and two sets of Higgs hypermultiplets, $\{H_1, H_1^c; H_2, H_2^c\}$ with $H_1, H_2 \in \mathbf{5}$ and $H_1^c, H_2^c \in \bar{\mathbf{5}}$. The parity under $y \to -y$ is given by

$$\mathscr{P}: \begin{pmatrix} V \\ \Sigma \end{pmatrix}(x_\mu, y, \theta) \to \begin{pmatrix} V \\ \Sigma \end{pmatrix}(x_\mu, -y, \theta) = \begin{pmatrix} V \\ -\Sigma \end{pmatrix}(x_\mu, y, \theta) \tag{17.1}$$

$$\begin{pmatrix} H_1 & H_2 \\ H_1^{c\dagger} & H_2^{c\dagger} \end{pmatrix}(x_\mu, y, \theta) \to \begin{pmatrix} H_1 & H_2 \\ H_1^{c\dagger} & H_2^{c\dagger} \end{pmatrix}(x_\mu, -y, \theta)$$

$$= \begin{pmatrix} H_1 & -H_2 \\ -H_1^{c\dagger} & H_2^{c\dagger} \end{pmatrix}(x_\mu, y, \theta).$$

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Note, in the Higgs sector we have an $SU(2)_H$ symmetry rotating $H_1 \leftrightarrow H_2$. We choose the holonomy under the shift symmetry to be an element of $U(1)_R \times U(1)_H$. We have $T_R = e^{-2\pi i\alpha\sigma_2}$ and $T_H = e^{2\pi i\gamma\sigma_2}$ where $\alpha \sim \gamma \sim 10^{-13}$ such that $\alpha/R \sim \gamma/R$ are of order the weak scale. In addition we have the GUT breaking holonomy, $T = (+ + + - -) \in SU(5)$. We then have

$$\mathscr{T}: \begin{pmatrix} A^M \\ \phi \end{pmatrix}(x_\mu, y) \to \begin{pmatrix} A^M \\ \phi \end{pmatrix}(x_\mu, y + 2\pi R) = \begin{pmatrix} T\, A^M\, T \\ T\, \phi\, T \end{pmatrix}(x_\mu, y) \tag{17.2}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}(x_\mu, y) \to \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}(x_\mu, y + 2\pi R) = T_R \begin{pmatrix} T\, \lambda_1\, T \\ T\, \lambda_2\, T \end{pmatrix}(x_\mu, y)$$

$$\begin{pmatrix} h_1 & h_2 \\ h_1^{c\dagger} & h_2^{c\dagger} \end{pmatrix}(x_\mu, y) \to \begin{pmatrix} h_1 & h_2 \\ h_1^{c\dagger} & h_2^{c\dagger} \end{pmatrix}(x_\mu, y + 2\pi R)$$

$$= T_R \begin{pmatrix} T\, h_1 & T\, h_2 \\ T\, h_1^{c\dagger} & T\, h_2^{c\dagger} \end{pmatrix}(x_\mu, y)\, T_H$$

$$\begin{pmatrix} \tilde{h}_1 & \tilde{h}_2 \\ \tilde{h}_1^{c\dagger} & \tilde{h}_2^{c\dagger} \end{pmatrix}(x_\mu, y) \to \begin{pmatrix} \tilde{h}_1 & \tilde{h}_2 \\ \tilde{h}_1^{c\dagger} & \tilde{h}_2^{c\dagger} \end{pmatrix}(x_\mu, y + 2\pi R)$$

$$= \begin{pmatrix} T\, \tilde{h}_1 & T\, \tilde{h}_2 \\ T\, \tilde{h}_1^{c\dagger} & T\, \tilde{h}_2^{c\dagger} \end{pmatrix}(x_\mu, y)\, T_H.$$

The fields in the MSSM with $P,\ P'(= TPP_F) = (++)$ (where the parity of Higgs fields under P_F is -1) are the only ones with zero modes (ignoring the SUSY breaking effects). These include the MSSM gauge sector, $V(++)$, and the Higgs doublets in $H_u = H_1(++),\ H_d = H_2^c(++)$. All other fields have only massive KK modes. When one includes the $T_R,\ T_H$ holonomies, then the mode expansion for the gauginos and Higgses are given by

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}(x_\mu, y) = \sum_{n=0}^{\infty} e^{-i\alpha\sigma_2 y/R} \begin{pmatrix} \lambda_{1n}(x_\mu)\cos[ny/R] \\ \lambda_{2n}(x_\mu)\sin[ny/R] \end{pmatrix}$$

$$\begin{pmatrix} h_1 & h_2 \\ h_1^{c\dagger} & h_2^{c\dagger} \end{pmatrix}(x_\mu, y) = \sum_{n=0}^{\infty} e^{-i\alpha\sigma_2 y/R} \begin{pmatrix} h_{1n}(x_\mu)\cos[ny/R] & h_{2n}(x_\mu)\sin[ny/R] \\ h_{1n}^{c\dagger}(x_\mu)\sin[ny/R] & h_{2n}^{c\dagger}(x_\mu)\cos[ny/R] \end{pmatrix} e^{i\gamma\sigma_2 y/R}$$

$$\begin{pmatrix} \tilde{h}_1 & \tilde{h}_2 \\ \tilde{h}_1^{c\dagger} & \tilde{h}_2^{c\dagger} \end{pmatrix}(x_\mu, y) = \sum_{n=0}^{\infty} \begin{pmatrix} \tilde{h}_{1n}(x_\mu)\cos[ny/R] & \tilde{h}_{2n}(x_\mu)\sin[ny/R] \\ \tilde{h}_{1n}^{c\dagger}(x_\mu)\sin[ny/R] & \tilde{h}_{2n}^{c\dagger}(x_\mu)\cos[ny/R] \end{pmatrix} e^{i\gamma\sigma_2 y/R}.$$

Substituting the mode expansions into the 5D action and integrating out the heavy modes one obtains the 4D effective field theory below the compactification scale. It contains the MSSM gauge and Higgs sectors. In addition, it contains the following mass terms

$$\mathscr{L} = -\frac{1}{2}\frac{\alpha}{R}(\lambda_0^a\lambda_0^a + h.c.) - \left(\frac{\alpha^2}{R^2} + \frac{\gamma^2}{R^2}\right)(|h_u|^2 + |h_d|^2) \qquad (17.3)$$
$$+2\frac{\alpha\gamma}{R^2}(h_u h_d + h.c.) - \frac{\gamma}{R}(\tilde{h}_u\tilde{h}_d + h.c.).$$

These are the soft SUSY breaking mass terms for gauginos, $M_{1/2} = \frac{\alpha}{R}$, Higgses, $m_{h_u}^2 = m_{h_d}^2 = \frac{\alpha^2+\gamma^2}{R^2}$, and $B\mu = \frac{2\alpha\gamma}{R^2}$ with the supersymmetric μ term given by $\mu = -\frac{\gamma}{R}$. Similar soft SUSY breaking mass terms can be obtained for squarks and sleptons, IF placed in the 5D bulk.

Gaugino Mediation

Consider a 5D orbifold GUT with the gauge and Higgs multiplets placed in the 5D bulk and the matter multiplets placed on the fixed point at $y = 0$ [333–335]. On the $y = \pi R$ fixed point we can locate the soft SUSY breaking sector of the theory with a chiral field ϕ_{hid} which obtains a non-vanishing F term, F_{id}. Then at zeroth order, SUSY breaking is transmitted to only the gauge and Higgs sectors of the theory, generating effective gaugino and Higgs terms of the form

$$\Delta\mathscr{L}_{brane} \sim \frac{\ell_5}{\ell_4}\{\left(\int d^2\theta\frac{\phi_{hid}}{M}W^\alpha W_\alpha + h.c.\right) + \left(\int d^4\theta\frac{\phi_{hid}^\dagger}{M^2}H_u H_d + h.c.\right)$$
$$+\int d^4\theta\frac{\phi_{hid}^\dagger\phi_{hid}}{M^3}(H_u^\dagger H_u + H_d^\dagger H_d + H_u H_d + h.c.)\} \qquad (17.4)$$

where M is the cut-off and $\ell_D = 2^D\pi^{D/2}\Gamma(D/2)$ is a geometrical loop factor for D dimensions.[1] In addition there are expected to be direct interactions between the hidden sector brane at $y = \pi R$ and the visible sector brane at $y = 0$ of the form

$$\Delta\mathscr{L}_{brane} \sim \frac{\ell_5}{\ell_4}e^{-M\pi R}\int d^4\theta\frac{\phi_{hid}^\dagger\phi_{hid}}{M^2}\phi_{obs}^\dagger\phi_{obs}. \qquad (17.5)$$

These are suppressed by the factor $e^{-\pi M/M_c} = e^{-ML}$ where $M_c = 1/R$ is the compactification scale and $L = \pi R$ is the volume of the extra dimension.

[1]The factor $\frac{\ell_5}{\ell_4}$ are supposed to take into account the relative size of loop corrections to the 4D vs. 5D Lagrangian.

Note, we have[2] $g^2 = g_5^2/L = \frac{\epsilon\, \ell_5}{L\, M}$ and the gauge field strength, W^α, and the Higgs fields, $H_{u,d}$ are normalized with canonical kinetic terms in 5D. Thus the correctly normalized Higgs fields in 4D are given by $H_{u,d}^{5D} = \frac{H_{u,d}^{4D}}{L^{1/2}}$. Hence when the F term of the hidden sector field, F_{hid}, is non-zero we obtain the soft SUSY breaking mass terms and μ term given by

$$M_{1/2},\ \mu \sim \frac{F_{hid}}{M}\frac{1}{LM}, \quad B\mu,\ m_{H_u}^2,\ m_{H_d}^2 \sim \left(\frac{F_{hid}}{M}\right)^2 \frac{1}{LM}. \tag{17.6}$$

We also obtain direct soft SUSY breaking mass terms for visible sector fields given by

$$\Delta m_{vis}^2 \sim e^{-M\,L} \left(\frac{F_{hid}}{M}\right)^2 \tag{17.7}$$

from the contact term of Eq. (17.5). These can induce flavor violating effects and thus should be suppressed. Given $g^2 \sim \epsilon \sim 1$, we have $L\,M \sim 20$ and we obtain a suppression by the factor $M\,L\,e^{-M\,L} \sim 4 \times 10^{-8}$ which is significant. The more important contribution to visible sector soft SUSY breaking masses comes from gaugino loops which are expected to generate scalar masses and A terms of order

$$\Delta m_{vis}^2 \sim \frac{g^2}{16\pi^2} M_{1/2}^2, \quad \Delta A_{vis} \sim \frac{g^2}{16\pi^2} M_{1/2}. \tag{17.8}$$

To summarize, the GUT scale boundary conditions expected from gaugino mediation are given by

- Gaugino masses: $M_1 = M_2 = M_3 = M_{1/2}$,
- Higgs masses: $m_{H_u}^2,\ m_{H_d}^2 \sim M_{1/2}^2,\ \ \mu,\ B \sim M_{1/2}$,
- Squark and slepton masses: $m^2 \sim \frac{M_{1/2}^2}{16\pi^2}$,
- A terms: $A \sim \frac{M_{1/2}}{16\pi^2}$.

Anomaly Mediation

The last transmission mechanism for SUSY breaking is known as anomaly mediation and was proposed in [336–338]. In anomaly mediation the hidden and visible sectors are again on separate branes, but now neither the gauge or Higgs sectors overlap both branes. In the terminology of [336], the hidden sector brane is

[2]By Naive Dimensional Analysis [NDA] it is assumed that the effective 5D gauge coupling is given by $g_5^2 \sim \frac{\epsilon\, \ell_5}{M}$ with $\epsilon \sim 1$ corresponding to strong coupling.

sequestered from the visible brane. Thus supersymmetry is broken on the hidden sector brane, but locally on the visible brane SUSY is preserved. Following the discussion of [336], let us consider the general Lagrangian for supergravity coupled to matter given by (in flat superspace notation)

$$\mathscr{L} = \sqrt{-g}\{\int d^4\theta\, f(Q^\dagger,\, e^{-V}\, Q)\, \Phi^\dagger\, \Phi + \int d^2\theta(\Phi^3\, \mathscr{W}(Q) + \tau(Q)\, \mathscr{W}^\alpha \mathscr{W}_\alpha) + h.c.$$
$$-\frac{1}{6} f(\tilde{q}^\dagger,\, \tilde{q})(R + \text{vector auxiliary terms + gravitino terms })\}, \tag{17.9}$$

where Q is a chiral superfield for visible matter with scalar component, $\tilde{q}$, and the vector superfield, V, with field strength, $\mathscr{W}_\alpha$. R is the Ricci curvature scalar and τ is the gauge kinetic function. The field Φ is a flat space chiral auxiliary field, in the Bagger and Wess supergravity formalism [120], given by

$$\Phi = 1 + F_\Phi\, \theta^2. \tag{17.10}$$

The function

$$f \equiv -3\, m_{Pl}^2\, e^{-\mathscr{K}/3m_{Pl}^2}, \tag{17.11}$$

where $\mathscr{K}$ is the Kähler potential.

The supergravity Lagrangian in the above Jordan form has a field dependent coefficient of the Ricci scalar curvature, R. By making a Weyl transformation of the metric, i.e.

$$g_{\mu\nu} \to e^{\mathscr{K}/3m_{Pl}^2}\, g_{\mu\nu} \tag{17.12}$$

and integrating out auxiliary fields one obtains

$$\mathscr{L} = \sqrt{-g}\{\frac{m_{Pl}^2}{2} R + \mathscr{K}_{\bar{i}j}(\tilde{q}^\dagger,\, \tilde{q})\, D_\mu \tilde{q}^{i\dagger}\, D^\mu \tilde{q}^j - V(\tilde{q}^\dagger,\, \tilde{q}) \tag{17.13}$$
$$-\tau(\tilde{q})(F_{\mu\nu}\, F^{\mu\nu} + iF_{\mu\nu}\, \tilde{F}^{\mu\nu}) + h.c. + \text{ fermion terms}\}$$

where

$$V = e^{\mathscr{K}/m_{Pl}^2}\{(\frac{\partial \mathscr{W}}{\partial \tilde{q}^i} + \frac{\mathscr{W}}{m_{Pl}^2}\frac{\partial \mathscr{K}}{\partial \tilde{q}^i})\, \mathscr{K}^{-1^{\bar{i}j}}\, (\frac{\partial \mathscr{W}^\dagger}{\partial \tilde{q}^{j\dagger}} + \frac{\mathscr{W}^\dagger}{m_{Pl}^2}\frac{\partial \mathscr{K}}{\partial \tilde{q}^{j\dagger}}) - 3\frac{|\mathscr{W}|^2}{m_{Pl}^2}\} \tag{17.14}$$
$$+\frac{g^2}{2}\, (\frac{\partial \mathscr{K}}{\partial \tilde{q}} T^A \tilde{q})^2.$$

Note, the solution of the Φ auxiliary field is given by

$$F_{\Phi}^{*} = \frac{\mathcal{W}}{m_{Pl}^2}\, e^{\mathcal{K}/3m_{Pl}^2}. \tag{17.15}$$

Assuming that the visible sector and hidden sectors are located on distant branes, we have

$$\begin{aligned} f &= -3m_{Pl}^2 + f_{vis} + f_{hid}, \\ \mathcal{W} &= \mathcal{W}_{vis} + \mathcal{W}_{hid}, \\ \tau\, \mathcal{W}^2 &= \tau_{vis}\, \mathcal{W}_{vis}^2 + \tau_{hid}\, \mathcal{W}_{hid}^2 \end{aligned} \tag{17.16}$$

where the visible and hidden sector functions only depend on the respective fields. Supersymmetry breaking is assumed to occur in the hidden sector via the F term of a superfield, Σ, with expectation value

$$\langle \Sigma \rangle = \sigma + \theta^2\, \Lambda_H^2. \tag{17.17}$$

In the flat space limit, once supersymmetry is broken, then the gravitino mass is given by $m_{3/2} = e^{\mathcal{K}/2m_{Pl}^2}\, \frac{|\mathcal{W}|}{m_{Pl}^2}$. But

$$\frac{F_{\Sigma}}{m_{Pl}} \sim \frac{\mathcal{W}}{m_{Pl}^2} \sim F_{\Phi} = \frac{\Lambda_H^2}{m_{Pl}}. \tag{17.18}$$

Therefore, the auxiliary field F_{Φ} also obtains a SUSY breaking VEV. Nevertheless, visible sector fields don't feel SUSY breaking at the tree level. This is because the auxiliary field Φ can be rescaled away by redefining the visible sector field such that

$$Q\Phi \to Q. \tag{17.19}$$

However, although this redefinition of the superfield Q is valid classically, it is violated by quantum corrections. In fact, this transformation corresponds to a scale transformation and even though the theory may be classically scale invariant, in order to renormalize the theory a renormalization scale, μ, must necessarily be defined. Thus, for example, at tree level we have the gauge coupling given by

$$\tau_0 = \frac{1}{g_0^2} \tag{17.20}$$

but after renormalization we have

$$\frac{1}{g^2} = \tau\left(\frac{\mu}{\Lambda_{UV}\Phi}\right) = \frac{1}{g_0^2} + 2b_0 \ln\left(\frac{\mu}{\Lambda_{UV}\Phi}\right) \tag{17.21}$$

such that the beta function for the gauge coupling, g, is given by

$$\beta(g) \equiv \frac{dg}{d\ln\mu} = -b_0\, g^3 + \ldots \tag{17.22}$$

with $b_0 = (3C_2(G) - T_R)/16\pi^2$ at one loop. Placing the renormalized gauge kinetic function τ back into the supergravity Lagrangian, Eq. (17.13) and using Eq. (17.10), we find a gaugino mass term given by[3]

$$M_{1/2} = -b_0\, g^2\, m_{3/2}. \tag{17.23}$$

The anomaly mediated contribution to gaugino masses in [338] allows for Planck scale expectation values in the hidden sector. Here the authors find corrections to Eq. (17.23) given by

$$M_{1/2} = -\frac{g^2}{16\pi^2}\left[(3C_2(G) - T_R)m_{3/2} + (C_2(G) - T_R)\mathcal{K}_i\, F^i + \frac{2T_R}{d_R}(\log\det\mathcal{K}|_R'')_i\, F^i\right] \tag{17.24}$$

where $\mathcal{K}$ is the Kähler potential and *prime* indicates derivatives with respect to visible-sector fields restricted to the representation, R, with dimension, d_R. A sum over all matter fields in representations R is implicit.

$$F^i = -e^{\mathcal{K}/2}\mathcal{K}^{ij^*}(\mathcal{W}_j + \mathcal{K}_j\,\mathcal{W})^*. \tag{17.25}$$

Now let's assume the sequestered form of the Kähler potential

$$\mathcal{K} = -3\log\left[-\frac{1}{3}Q^\dagger\, Q + f(H^\dagger, H)\right] \tag{17.26}$$

with Q and H denoting visible and hidden-sector chiral superfields, respectively. Moreover, assuming that the VEVs of visible-sector fields vanish we have

$$\frac{1}{d_R}\log\det\mathcal{K}|_R'' = \frac{1}{3}\mathcal{K}. \tag{17.27}$$

We then find

$$M_{1/2} = -\frac{g^2}{16\pi^2}\,(3C_2(G) - T_R)\left(m_{3/2} + \frac{1}{3}\mathcal{K}_i\, F^i\right). \tag{17.28}$$

[3]The gauge kinetic term is usually defined with a factor of 1/4 which then gives the standard normalization of the gauge coupling constant.

As an example of the possible effect of the additional term, consider a "no-scale" model, with Kähler potential

$$\mathscr{K} = -3\log\left[T + T^\dagger - \frac{1}{3}Q^\dagger\, Q - \frac{1}{3}H^\dagger\, H)\right] \tag{17.29}$$

with the modulus[4] T in the hidden-sector. If supersymmetry breaking is dominated by the F component of T, and a constant is added to the superpotential to cancel the cosmological constant, we have

$$\mathscr{K}_T\, F^T = -3\, m_{3/2} \tag{17.30}$$

and the anomaly-induced gaugino mass vanishes. With more than one supersymmetry breaking VEV near the Planck scale the anomaly-induced gaugino masses might be suppressed.

Scalar masses are also generated at two loops. The Wilsonian effective Lagrangian (i.e. obtained by integrating out all momentum from μ to Λ_{UV}) is given by

$$\begin{aligned}\mathscr{L}_{\mathit{eff}} = & \int d^4\theta Z\left(\frac{\mu}{\Lambda_{UV}\Phi},\ \frac{\mu}{\Lambda_{UV}\Phi^\dagger}\right)\, Q^\dagger\, e^{-V}\, Q \\ & + \int d^2\theta\ (Y_0\, Q^3 + \tau\left(\frac{\mu}{\Lambda_{UV}\Phi}\right)\, \mathscr{W}_\alpha^2) + h.c..\end{aligned} \tag{17.31}$$

Now using

$$\begin{aligned}\ln Z\left(\frac{\mu}{\Lambda_{UV}(\Phi\ \Phi^\dagger)^{1/2}}\right) = & \ln Z\left(\frac{\mu}{\Lambda_{UV}}\right) - \frac{1}{2}F_\Phi\theta^2\frac{d\ln Z}{d\ln\mu}(\frac{\mu}{\Lambda_{UV}}) + h.c. \\ & + \frac{1}{4}|F_\Phi|^2\theta^2\bar{\theta}^2\ \frac{d^2\ln Z}{d(\ln\mu)^2}(\frac{\mu}{\Lambda_{UV}})\end{aligned} \tag{17.32}$$

or

$$\begin{aligned}\ln Z\left(\frac{\mu}{\Lambda_{UV}|\Phi|}\right) \equiv & \ln Z\left(\frac{\mu}{\Lambda_{UV}}\right) - \frac{1}{2}\,\gamma(g,\ Y)\ (F_\Phi\theta^2 + h.c.) \\ & + \frac{1}{4}|F_\Phi|^2\theta^2\bar{\theta}^2\ \left(\frac{\partial\gamma}{\partial g}\beta_g + \frac{\partial\gamma}{\partial Y}\beta_Y\right),\end{aligned} \tag{17.33}$$

[4]Perhaps T is a volume modulus determining the size of the extra dimension.

where we used the renormalization group functions

$$\gamma(g,\ Y) \equiv \frac{\partial \ln Z}{\partial \ln \mu},\quad \beta_g \equiv \frac{\partial g}{\partial \ln \mu},\quad \beta_Y(g,\ Y) \equiv \frac{\partial Y}{\partial \ln \mu}. \tag{17.34}$$

The linear terms in F_Φ can be scaled away by the redefinition

$$exp\{\frac{1}{2}\ln Z(\frac{\mu}{\Lambda_{UV}}) - \frac{1}{2}\gamma(g,\ Y)F_\Phi\theta^2\}Q \to Q \tag{17.35}$$

leaving only the quadratic terms in F_Φ. We then have

$$Z\left(\frac{\mu}{\Lambda_{UV}|\Phi|}\right) = 1 + \frac{1}{4}\ |F_\Phi|^2\theta^2\bar{\theta}^2\ \left(\frac{\partial\gamma}{\partial g}\beta_g + \frac{\partial\gamma}{\partial Y}\beta_Y\right). \tag{17.36}$$

Again, plugging in $\Phi = 1 + F_\Phi\ \theta^2$ or $\ln\Phi = F_\Phi\ \theta^2$ and performing the θ integrals we find the scalar mass squared

$$m_{\tilde{q}}^2(\mu) = -\frac{1}{4}|F_\Phi|^2\left(\frac{\partial\gamma}{\partial g}\beta_g + \frac{\partial\gamma}{\partial Y}\beta_Y\right). \tag{17.37}$$

This is a two loop result since the anomalous dimensions, γ and the beta functions $\beta_g,\ \beta_Y$ are all one loop results. Using the general forms

$$\gamma = c_0 g^2 + d_0 Y^2,\quad \beta_g = -b_0 g^3,\quad \beta_Y = Y(e_0 Y^2 + f_0 g^2) \tag{17.38}$$

we obtain the scalar mass squared

$$m_{\tilde{q}}^2 = \frac{1}{2}\{c_0 b_0 g^4 - d_0 Y^2(e_0 Y^2 + f_0 g^2)\}|F_\Phi|^2. \tag{17.39}$$

Note, the constant $c_0 > 0$ and $b_0 > 0$ for asymptotically free gauge theories, but $b_0 < 0$ for infra-red free gauge theories. Thus in the MSSM, since right-handed sleptons only have $U(1)_Y$ gauge interactions, their mass squared is negative. This is a problem which however can be fixed with additional interactions, such as D terms from additional low energy $U(1)$ gauge interactions, see [339, 340].

Finally, cubic scalar interactions are generated from the re-scaled cubic terms. These generate A terms given by

$$A_{ijk} = \frac{1}{2}(\gamma_i + \gamma_j + \gamma_k)\ Y_{ijk}\ F_\Phi. \tag{17.40}$$

Mirage Mediation and Precise Gauge Coupling Unification

In general orbifold GUT models, it is possible to have several different SUSY breaking mechanisms working simultaneously. For example, it is possible that the dominant SUSY breaking effect gives visible sector scalars a large mass of order the gravitino mass, $m_{3/2} \sim \frac{F}{m_{Pl}}$. However there are additional SUSY breaking VEVs, F', which give gauginos mass, but suppressed by loop effects. This happens because the SUSY breaking F term only enters the gauge kinetic function at one loop. We then obtain $M_{1/2} \sim \frac{F'}{16\pi^2 m_{Pl}}$. It is also possible that the contribution to the gaugino mass due to anomaly mediation can be of the same order.[5] We then have mirage mediation, Eq. (10.5), with

$$M_i = \left(1 - \frac{g_G^2 b_i \alpha}{16\pi^2} \log\left(\frac{M_{Pl}}{m_{16}}\right)\right) M_{1/2} \,. \tag{17.41}$$

It has been shown that if gluino masses are lighter than obtained in the case of universal gaugino masses, see [198, 341], that precision gauge coupling unification [PGCU] is possible, i.e. $\epsilon_3 = 0$. In PGCU it is also possible to have a well-tempered dark matter candidate [198, 238]. Thus the question arises, what physics at the GUT scale naturally gives $\epsilon_3 = 0$? In the next section, we show that PGCU can be obtained with non-local Wilson line symmetry breaking. In this case, all gauge couplings unify (up to small corrections) at the compactification scale. Said another way, if the GUT symmetry is broken via non-local Wilson lines, then the low energy theory requires either mirage mediation or general gauge mediation with lighter gluinos!

[5] Of course, there could also be additional SUSY breaking contributions to gaugino masses, such as gauge-mediated terms.

Chapter 18
SUSY GUTs in 6D: Precise Gauge Coupling Unification

In the previous chapter we argued that precise gauge coupling unification [PGCU], i.e. the absence of threshold corrections at the GUT scale, requires certain specific soft SUSY breaking masses at the weak scale. In particular, lighter gluinos than expected from unification of gaugino masses. In this chapter we argue that non-local GUT breaking will, in general, eliminate one-loop GUT threshold corrections. This is because the compactification scale is the scale of GUT symmetry breaking [342–344]. In this chapter (based on [345]) we present a 6D model with $SU(6)$ gauge symmetry and N=2 supersymmetry. In terms of 4D language, such a 6D theory with N=2 SUSY contains one vector adjoint and three chiral adjoints. The model has gauge-Higgs unification with the Higgs doublets coming from one of the chiral adjoints. The group $SU(6)$ is broken to $SU(5) \times U(1)_X$ via orbifold boundary conditions. Then $SU(5)$ is broken to the Standard Model gauge group and, at the same time, Higgs doublet-triplet splitting is accomplished by a Wilson line. The two extra-dimensions are compactified on an orbifold that can be characterized as a sphere with a cross-cap (topologically equivalent to the manifold RP^2), as described in [342–344]. Quarks and leptons can be in the bulk (or localized at the fixed points), and their Yukawa couplings to the Higgs are localized at the orbifold fixed points which only retain an N=1 SUSY in 4D (see for example [313, 346] where this phenomenon has been discussed).

It is worthwhile to explain why we choose to compactify the two extra spatial dimensions on RP^2. It has to do with GUT symmetry breaking via Wilson lines. In previous chapters in 5D we compactified the extra dimension on a line segment $y \in [0, \pi R]$. For example, in the case of $SU(5)$, the discrete Wilson line breaks $SU(5)$ spontaneously to the SM gauge group, $SU(3) \times SU(2) \times U(1)$. However the gauge symmetry on the $y = \pi R$ brane is locally broken to the SM. We can also consider a 6D GUT where we compactify the extra two dimensions on the two dimensional surface of a 3 dimensional sphere. However, a Wilson line on this surface can be shrunk to a point, i.e. $\Pi_1(S^2) = 0$, the

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manifold is simply connected. In order to avoid this possibility let's define a space which is topologically equivalent to the surface of the sphere, but has a non-trivial Π_1. Such a two dimensional surface is RP^2 which can be obtained, without orbifolding, by taking the surface of a 2 sphere and identifying anti-podal points. Such a space can also be obtained by orbifolding. We first define the 2D surface as a torus mod a $\mathbb{Z}_2$ symmetry. The 2 torus T^2 is defined by taking R^2 with coordinates $x_5,\ x_6$ and mod out by the discrete translations, $\mathscr{T}_5,\ \mathscr{T}_6$ such that

$$\mathscr{T}_i : x_i \to x_i + 2\pi R_i, \quad i = 5, 6. \tag{18.1}$$

Then the $\mathbb{Z}_2$ action is given by

$$\mathbb{Z}_2 : x_i \to -x_i, \quad i = 5, 6. \tag{18.2}$$

This is the first step described in Fig. 18.1. The end result is an orbifold, topologically equivalent to S^2, can be described as the surface of a pillow with four fixed points. We can now place a Wilson line in either the five or six directions and use it to break $SU(5)$ to the SM. Once again the gauge symmetry is explicitly broken to the SM at one or more of the fixed points. In the case of either the 5D or 6D example we described, the gauge couplings unify at the cut-off scale with logarithmic running of differences of the gauge couplings between the compactification scale and the cut-off, i.e. unification only occurs at the cut-off. Both of these cases are examples of local GUT breaking. On the other hand, the manifold RP^2, or its orbifold equivalent, is a manifold with $\Pi_1(RP^2) = \mathbb{Z}_2$. This means that a Wilson line can traverse the manifold along a closed circle and still not be continuously deformed to a point. As a consequence, this Wilson line will break the gauge symmetry non-locally on the entire manifold.[1] As a result we expect that the compactification scale can be identified with the 4D GUT scale, since we expect to find precise gauge coupling unification [PGCU] at M_c. The purpose of this section is to show that indeed, this is the case.

The details of the orbifold and the symmetry breaking are discussed in Sect. 18.1. We break the $SU(6) \to SU(5) \times U(1)_X$ using one of the orbifold projections, locally at the fixed points. We then break the $SU(5) \to SU(3) \times SU(2) \times U(1)_Y$ using a Wilson line along the fifth and sixth directions. In Sect. 18.5, we analyze gauge coupling unification in the SU(6) GUT model constructed on such an orbifold and calculate the GUT-scale threshold corrections in this scenario. We find that unlike in most popular models of orbifold GUTs, the couplings do not receive power law corrections at any scale and the logarith-

[1] Wilson line breaking on a smooth RP^2 is certainly non-local breaking. On the orbifold version of RP^2 we shall see that the Wilson line breaking is not completely non-local, although the results, as we shall show, are consistent with non-local breaking of the GUT symmetry.

mic corrections above the largest compactification scale are $SU(5)$ invariant, both effects due to the effective N=4 SUSY in 4D. We analyze the GUT-scale threshold corrections to determine if they are at the required level to match low energy physics. We point out that an example of an orbifold GUT from a 6D $SU(6)$ was considered in [346] with the similar feature of gauge-Higgs unification. The extra-dimensions were compactified on $T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2')$ and the authors obtain realistic phenomenology with local GUT breaking. The 6D GUT theory also had an N=2 supersymmetry. As a consequence, the coefficient of the effective 6D quadratic power law dependence of the gauge couplings vanished. But due to the existence of fixed lines the effective 5D linear dependence remained.

18.1 GUT Breaking

Real Projective Plane

An N=2 supersymmetric $SU(6)$ gauge theory in six dimensions is compactified on an orbifold, shown in Fig. 18.1, as described in Hebecker [342]. The extra dimensions are compactified on a torus T^2 parametrized by (x_5, x_6). The two dimensions are also identified to have the periodicity, $x_{(5,6)} = x_{(5,6)} + 2\pi R_{(5,6)}$, where R_5 and R_6 are the radius of the torus along the two directions. Two discrete symmetries, the rotation $\mathscr{Z}$ and a freely acting roto-translation $\mathscr{Z}'$, as defined in Eqs. (18.3), (18.4) are modded out. Once the first symmetry is modded out, the topology of the compact space is that of a 2-sphere with curvature concentrated at the four conical singularities. The space resembles a pillow with fundamental group $\Pi_1 = \emptyset$. Once the second parity is modded out, the resulting compact space is equivalent to a projective plane, RP^2. It

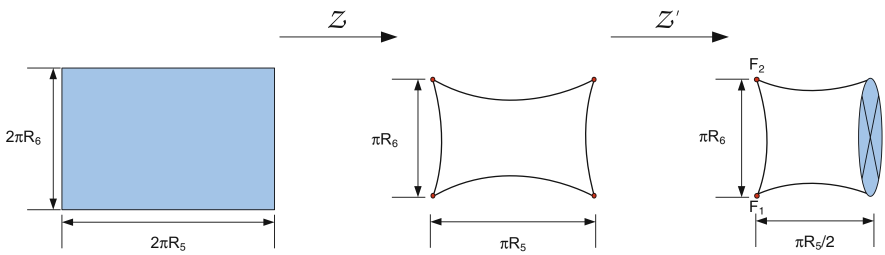

Fig. 18.1 The figure shows the manifold at each step of the compactification. After the first step of orbifolding, the space looks like a pillow with four fixed points denoted by *red dots* in the center figure. After the second step of orbifolding as described in [342], this space is equivalent to a real projective plane. *Reprinted from Nuclear Physics B 868, A. Anandakrishnan and S. Raby, "SU(6) GUT breaking on a projective plane," Page 629, Copyright (2013), with permission from Elsevier. This right-hand figure was reproduced from Fig. 1, [342]*

is non-orientable with no boundaries, the curvature is concentrated at the two fixed points denoted by F_1 and F_2 and $\Pi_1 = \mathbb{Z}_2$. The non-orientability of the space can be ascribed to the cross-cap where opposite points on the circle are identified.[2]

$$\mathscr{Z} \qquad x_5 \to -x_5, \qquad x_6 \to -x_6 \tag{18.3}$$

$$\mathscr{Z}' \qquad x_5 \to -x_5 + \pi R_5, \qquad x_6 \to x_6 + \pi R_6. \tag{18.4}$$

We choose to write the particle content of the theory in terms of the 4D language. There is one vector superfield, V and three chiral superfields, Σ_5, Σ_6, and Φ; all in the adjoint representation of $SU(6)$. Using the notation in [346], the bulk action in the Wess-Zumino gauge is given by:

$$\begin{aligned} S = \int d^6x \Bigg\{ \mathrm{Tr}\Bigg[\int d^2\theta \Bigg(\frac{1}{4kg^2} \mathscr{W}^\alpha \mathscr{W}_\alpha \\ + \frac{1}{kg^2}\Big(\Phi\partial_5\Sigma_6 - \Phi\partial_6\Sigma_5 - \frac{1}{\sqrt{2}}\Phi[\Sigma_5, \Sigma_6]\Big)\Bigg) + \text{h.c.}\Bigg] \\ + \int d^4\theta \frac{1}{kg^2} \mathrm{Tr}\Bigg[(\sqrt{2}\partial_5 + \Sigma_5^\dagger)e^{-V}(-\sqrt{2}\partial_5 + \Sigma_5)e^V \\ +(\sqrt{2}\partial_6 + \Sigma_6^\dagger)e^{-V}(-\sqrt{2}\partial_6 + \Sigma_6)e^V \\ +\Phi^\dagger e^{-V}\Phi e^V + \partial_5 e^{-V}\partial_5 e^V + \partial_6 e^{-V}\partial_6 e^V \Bigg]\Bigg\} \end{aligned} \tag{18.5}$$

18.2 $SU(6) \to SU(5) \times U(1)_X$

The 6D N=2 supersymmetric theory that we start with has an effective N=4 SUSY in 4 dimensions. The action of the above discussed parities can be used to break the gauge group $SU(6)$ down to $SU(5) \times U(1)_X$, and at the same time break N = 4 SUSY to N = 1 SUSY (in 4D) [335]. We can break $SU(6)$ to $SU(5) \times U(1)_X$ by requiring the fields to transform as illustrated below, under the two parities.

[2] It appears that fermions and supersymmetry can be defined on some non-orientable manifolds. See, for example, some general theorems on Pin manifolds [347–349] and some specific examples of fermions on $M_4 \otimes RP^2$ [350–352].

Under the parity, $\mathscr{Z}$:

$$\begin{aligned}
V(-x_5, -x_6) &= PV(x_5, x_6)P^{-1}, \\
\Sigma_5(-x_5, -x_6) &= -P\Sigma_5(x_5, x_6)P^{-1}, \\
\Sigma_6(-x_5, -x_6) &= -P\Sigma_6(x_5, x_6)P^{-1}, \\
\Phi(-x_5, -x_6) &= P\Phi(x_5, x_6)P^{-1},
\end{aligned} \tag{18.6}$$

Under the parity, $\mathscr{Z}'$:

$$\begin{aligned}
V(-x_5 + \pi R_5, x_6 + \pi R_6) &= V(x_5, x_6), \\
\Sigma_5(-x_5 + \pi R_5, x_6 + \pi R_6) &= -\Sigma_5(x_5, x_6), \\
\Sigma_6(-x_5 + \pi R_5, x_6 + \pi R_6) &= \Sigma_6(x_5, x_6), \\
\Phi(-x_5 + \pi R_5, x_6 + \pi R_6) &= -\Phi(x_5, x_6).
\end{aligned} \tag{18.7}$$

where P = diag$(i, i, i, i, i, -i)$ breaks the $SU(6) \to SU(5) \times U(1)_X$. The projection $\mathscr{Z}$ has four fixed points (as shown in Fig. 18.1). The $SU(6)$ symmetry is explicitly broken to $SU(5) \times U(1)_X$ at the three non-(0,0) fixed points. In the next section, we use a non-local Wilson line to further break $SU(5)$ to $SU(3) \times SU(2) \times U(1)_Y$.

Under the combined operation $(\mathscr{Z}, \mathscr{Z}')$ the components of the fields transform as follows:

$$V = \left(\begin{array}{ccc|cc|c}
(++) & (++) & (++) & (++) & (++) & (-+) \\
(++) & (++) & (++) & (++) & (++) & (-+) \\
(++) & (++) & (++) & (++) & (++) & (-+) \\ \hline
(++) & (++) & (++) & (++) & (++) & (-+) \\
(++) & (++) & (++) & (++) & (++) & (-+) \\ \hline
(-+) & (-+) & (-+) & (-+) & (-+) & (++)
\end{array}\right)$$

$$\Sigma_5 = \left(\begin{array}{ccc|cc|c}
(--) & (--) & (--) & (--) & (--) & (+-) \\
(--) & (--) & (--) & (--) & (--) & (+-) \\
(--) & (--) & (--) & (--) & (--) & (+-) \\ \hline
(--) & (--) & (--) & (--) & (--) & (+-) \\
(--) & (--) & (--) & (--) & (--) & (+-) \\ \hline
(+-) & (+-) & (+-) & (+-) & (+-) & (--)
\end{array}\right)$$

$$
\Sigma_6 = \left(\begin{array}{c|c|c}
(-+)(-+)(-+) & (-+)(-+) & (++) \\
(-+)(-+)(-+) & (-+)(-+) & (++) \\
(-+)(-+)(-+) & (-+)(-+) & (++) \\
\hline
(-+)(-+)(-+) & (-+)(-+) & (++) \\
(-+)(-+)(-+) & (-+)(-+) & (++) \\
\hline
(++)(++)(++) & (++)(++) & (-+)
\end{array}\right)
$$

$$
\Phi = \left(\begin{array}{c|c|c}
(+-)(+-)(+-) & (+-)(+-) & (--) \\
(+-)(+-)(+-) & (+-)(+-) & (--) \\
(+-)(+-)(+-) & (+-)(+-) & (--) \\
\hline
(+-)(+-)(+-) & (+-)(+-) & (--) \\
(+-)(+-)(+-) & (+-)(+-) & (--) \\
\hline
(--)(--)(--) & (--)(--) & (+-)
\end{array}\right) \tag{18.8}
$$

The parity operations $(\mathscr{Z}, \mathscr{Z}')$ performed on the coordinate space are symmetries of the Lagrangian, hence the fields in the Lagrangian must be eigenstates of the parity operations. A general field $\varphi = \{V, \Sigma_5, \Sigma_6, \Phi\}$ by definition of the manifold, are periodic functions of x_5 and x_6.

$$
\begin{aligned}
\varphi(x, x_5 + 2\pi R_5, x_6) &= \varphi(x, x_5, x_6) \\
\varphi(x, x_5, x_6 + 2\pi R_6) &= \varphi(x, x_5, x_6)
\end{aligned} \tag{18.9}
$$

This allows us to expand them as:

$$
\varphi(x, x_5, x_6) = \frac{1}{\sqrt{2\pi R_5 R_6}} \sum_{m,n=-\infty}^{+\infty} \varphi^{(m,n)} \exp\left[i \left(\frac{m x_5}{R_5} + \frac{n x_6}{R_6} \right) \right] \tag{18.10}
$$

The eigenstates of the parity operations are required to obey:

$$
\begin{aligned}
\varphi_{\pm\widehat{\pm}}(x_\mu, -x_5, -x_6) &= \pm \varphi_{\pm\widehat{\pm}}(x_\mu, x_5, x_6) \\
\varphi_{\pm\widehat{\pm}}(x_\mu, -x_5 + \pi R_5, x_6 + \pi R_6) &= \widehat{\pm} \varphi_{\pm\widehat{\pm}}(x_\mu, x_5, x_6)
\end{aligned} \tag{18.11}
$$

which project out even and odd modes that can be written out as:

$$
\begin{aligned}
\varphi_{\pm\widehat{\pm}}(x, x_5, x_6) = {} & \frac{1}{4\sqrt{2\pi R_5 R_6}} \sum_{m,n} \left[(\varphi^{(m,n)} \pm \varphi^{(-m,-n)}) \widehat{\pm} (-1)^{m-n} (\varphi^{(-m,n)} \pm \varphi^{(m,-n)}) \right] \\
& \times \exp\left[i \left(\frac{m x_5}{R_5} + \frac{n x_6}{R_6} \right) \right]
\end{aligned} \tag{18.12}
$$

In the above three expressions, $\pm$ denotes states that are even/odd under the first parity operation, and $\widehat{\pm}$ denotes states that are even/odd under the second parity. The

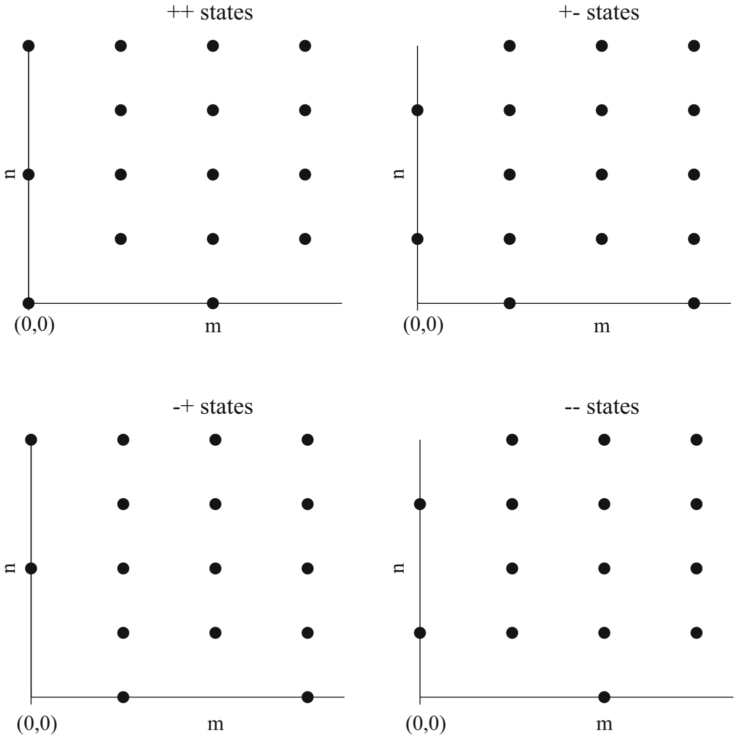

Fig. 18.2 The mode expansion in Eq. (18.12) gives the information about where the various parity eigenstates exist. Notice that this figure depicts only the positive parts of the (m,n) values while for the calculations they should be summed over both positive and negative integers. It is clear from the figure that only the $(++)$ fields have zero modes. *Reprinted from Nuclear Physics B 868, A. Anandakrishnan and S. Raby, "SU(6) GUT breaking on a projective plane," Page 632, Copyright (2013), with permission from Elsevier*

massless modes come only from the $+\widehat{+}$ (hereafter denoted as ++) parity modes. They appear in the adjoint representation of $SU(5) \times U(1)_X$ in V and in the $\mathbf{5}$ and $\bar{\mathbf{5}}$ of $SU(5)$ in Σ_6. These latter contain the MSSM Higgs bosons. The above spectrum is illustrated in Fig. 18.2.

18.3 $SU(5) \to SU(3) \times SU(2) \times U(1)_Y$

We now introduce a Wilson line to break the symmetry down to the Standard Model. A gauge field, $A_M \equiv \sum_a A^a_M T^a$ transforms under a gauge transformation as follows:

$$A_M(x_\mu, x_5, x_6) \to U A_M(x_\mu, x_5, x_6) U^\dagger - iU\partial_M U^\dagger \tag{18.13}$$

where T^a correspond to the generators of the gauge group.[3] Now consider a constant background gauge field along the fifth and sixth directions:

$$A_5 = \frac{1}{4R_5}T \quad \text{and,} \quad A_6 = \frac{1}{4R_6}T \tag{18.14}$$

where T is the generator (up to a constant) that breaks $SU(6)$ down to $SU(3) \times SU(3) \times U(1)$ given by:[4]

$$T = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix} \tag{18.15}$$

Note that the choice of the background gauge fields must obey some strict constraints. For example, the space group generators obey:

$$\mathscr{Z}^2 = \mathbb{1}, \qquad \mathscr{Z}'^2 = \mathscr{T}_6 \tag{18.16}$$

The second condition implies that the action of the parity $\mathscr{Z}'$ is equivalent to the holonomy coming from the gauge field along the sixth direction. In addition,

$$\mathscr{Z}\mathscr{Z}'\mathscr{Z}\mathscr{Z}' = \mathscr{T}_5^{-1} \tag{18.17}$$

Rewriting the above relation of the space group generators as holonomies, we get:

$$G(\mathscr{Z}^2)G(\mathscr{Z}'^2) = G(\mathscr{T}_5^{-1}) \tag{18.18}$$

where we have use the fact that U(1) holonomies commute. Noting that $G(\mathscr{T}_5^{-1}) = G(\mathscr{T}_5)$, we find that the holonomies should obey the condition:

$$G(\mathscr{T}_5) = G(\mathscr{T}_6) \tag{18.19}$$

This statement tells us that the Wilson lines cannot be independent along the two extra-dimensions.

---

[3]This is the remaining gauge symmetry of the supersymmetric theory in the Wess-Zumino gauge.

[4]This constant background field is consistent with the parity operation $A_5 \to -A_5$ with the additional periodic gauge transformation, such that $A_5' = U(x_5)(-A_5)U(x_5)^\dagger - iU(x_5)\partial_{x_5}U(x_5)^\dagger \equiv A_5$ and $U(x_5) = \exp\left(-i\frac{x_5}{R_5}\frac{T}{2}\right)$ is periodic under $x_5 \to x_5 + 2\pi R_5$ up to an element of the center of the group $SU(6)$ [291].

The presence of such a background gauge field breaks the gauge symmetry. The constant background fields introduce a holonomy equal to

$$W = \exp\left(i \oint A_5 dx_5 + i \oint A_6 dx_6\right). \tag{18.20}$$

This non-trivial holonomy affects the spectrum of Kaluza-Klein states. In an equivalent picture [291, 299], the background gauge field can be gauged away completely by choosing the proper gauge transformation, and in this case, we find that the gauge condensate vanishes when we choose to redefine fields by the gauge transformation

$$U(x_5) = \exp\left[i\left(\frac{x_5}{R_5} + \frac{x_6}{R_6}\right)\frac{T}{4}\right]. \tag{18.21}$$

Nevertheless, the physics remains unchanged, and we determine the change in the KK spectrum due to the non-trivial holonomy (or Wilson-line).

Under the gauge transformation operator, Eq. (18.21), a generic adjoint field φ transforms as:

$$\varphi'(x_\mu, x_5, x_6) = U(x_5, x_6)\varphi(x_\mu, x_5, x_6)U^\dagger(x_5, x_6) \tag{18.22}$$

which allows us to rewrite the gauge transformed wave function as

$$\varphi'(x_\mu, x_5, x_6) = \mathrm{e}^{i\left(\frac{x_5}{R_5} + \frac{x_6}{R_6}\right)\frac{I_\rho}{4}}\varphi(x_\mu, x_5, x_6) \tag{18.23}$$

where, I_ρ is the eigenvalue of the generator T and $\varphi(x_\mu, x_5, x_6)$ is the untransformed wave function as defined in Eq. (18.10). The periodicity condition Eq. (18.9) of the fields then becomes:

$$\begin{aligned}\varphi'(x_\mu, x_5 + 2\pi R_5, x_6) &= P'\varphi'(x_\mu, x_5, x_6)P'^\dagger \equiv e^{i\frac{\pi}{2}I_\rho}\varphi'(x_\mu, x_5, x_6)\\ \varphi'(x_\mu, x_5, x_6 + 2\pi R_6) &= P'\varphi'(x_\mu, x_5, x_6)P'^\dagger = e^{i\frac{\pi}{2}I_\rho}\varphi'(x_\mu, x_5, x_6)\end{aligned} \tag{18.24}$$

where $P' \equiv \exp\left(i\frac{\pi}{2}T\right) = \mathrm{diag}(i, i, i, -i, -i, -i)$.[5] The above equation reflects the constraints on the Wilson lines that was demonstrated in Eq. (18.19). In addition, now we have re-expressed the Wilson line as a parity operation that breaks $SU(6)$ down to $SU(3) \times SU(3) \times U(1)$. Under the combined parity operations on the manifold and the non-vanishing background fields along the fifth and sixth directions, we have achieved gauge symmetry breaking of the $SU(6)$ group to

[5] In our case we can now identify $G(\mathcal{T}_5) = G(\mathcal{T}_6) = e^{i\frac{\pi}{2}I_\rho}$. There is, however, an alternative possibility with $G(\mathcal{T}_5) = G(\mathcal{T}_6) = \mathbb{1}$ and $G(\mathcal{Z}'^2) = \mathbb{1}$. This is the possibility discussed in the paper by Hebecker [342]. This choice leads to additional massless vector-like exotics which we avoid in our analysis.

$[SU(3)\times SU(2)\times U(1)_Y]\times U(1)_X$. The $SU(5)$ GUT breaking that we have described, although via a Wilson line on a manifold with $\Pi(1) = \mathbb{Z}_2$, is local, due to our choice of holonomy on the torus. As a result there is non-trivial $U(1)$ gauge flux localized at the fixed point F_2 (Fig. 18.1). Thus the two fixed points have different local symmetries. F_1 is invariant under SU(5) × U$(1)_X$, while F_2 is only invariant under $[SU(3)\times SU(2)\times U(1)_Y]\times U(1)_X$.[6]

We still have to calculate how the mass spectrum changes as a result of the holonomy due to the gauge field. This can easily be done by looking at the transformed wave function in Eq. (18.23) and calculating the eigenvalues I_ρ of the generator T. The eigenvalues I_ρ can be determined by calculating the commutator $[T,\varphi]$ since φ is in the adjoint representation, of the form:

$$
\begin{aligned}
\varphi &= \left(\begin{array}{c|c|c}
(8,1)_0 & (3,\bar{2})_{-5/3} & (3,1)_{-2/3} \\
\hline
(\bar{3},2)_{5/3} & (1,3)_0 & (1,2)_1 \\
\hline
(\bar{3},1)_{2/3} & (1,\bar{2})_{-1} & (1,1)_0
\end{array}\right) \\
&= \left(\begin{array}{c|c|c}
g & X & T \\
\hline
\bar{X} & w & H_u \\
\hline
\bar{T} & H_d & b
\end{array}\right)
\end{aligned}
\tag{18.25}
$$

The first line in the above expression shows the quantum numbers of the different blocks that the adjoint field gets broken into after the orbifold projection and holonomy. We name them appropriately, so that they can be associated with the fields that remain massless in the low energy theory, like the gauge bosons, g, w, b and the Higgs doublets, H_u, H_d; and the fields that obtain mass and do not appear in the low energy spectrum like the Higgs triplets $T, \bar{T}$ and states with exotic quantum numbers $X, \bar{X}$. The commutator of the generator T with this quantity is calculated and the eigenvalues of are summarized in Table 18.1.

Eventually, we see that the masses of the states in the KK tower are given by

$$
M^2_{(m,n),\rho} = \frac{(m+\frac{I_\rho}{4})^2}{R_5^2} + \frac{(n+\frac{I_\rho}{4})^2}{R_6^2} \tag{18.26}
$$

[6] In what follows, we shall ignore any explicit gauge symmetry breaking localized at the fixed point, F_2. This might be due to terms such as $-\frac{1}{4g_a^2}\int d^4x F_a^{\mu\nu} F_{\mu\nu a}$, $a = 1, 2, 3$. In the orbifold field theory these will be suppressed by the volume of the extra dimensions, while in string theories these do not occur. Moreover, in a smooth RP_2 these would not exist.

Table 18.1 Eigenvalues I_ρ of the generator T acting on the various fields (labelled by ρ) in the model

	g	w	b	X	$\bar{X}$	T	$\bar{T}$	H_u	H_d
I_ρ	0	0	0	2	−2	2	−2	0	0

The massless states are those which are even under both parities and have zero eigenvalue under the holonomy. These turn out to be only the Standard Model gauge bosons in V and the Higgs doublets, H_u, H_d coming from the chiral adjoint Σ_6. The three families of quarks and leptons are also assumed to sit at either of the two fixed points. Any family sitting at F_1 will come in complete SU(5) multiplets $(\mathbf{10_F} + \mathbf{\bar{5}_F})$ representations, while at F_2, SU(5) relations are not preserved. The Yukawa couplings are also assumed to be localized at these fixed points. They require superpotential terms of the form $\mathbf{10}_F\ \mathbf{10}_F\ \mathbf{5}_{\Sigma_6} + \mathbf{10}_F\ \mathbf{\bar{5}}_F\ \mathbf{\bar{5}}_{\Sigma_6}$ where the indices are contracted in an obvious way. The SU(5) relation $\lambda_b = \lambda_\tau$ works for the third family but not for the first two, so the third family could be placed at F_1 with the first two sitting at F_2 or in the bulk. Finally, in order to obtain only the Standard Model gauge symmetry at low energies we need to introduce SU(5) singlets $\mathbf{1}^+ + \mathbf{1}^-$ with $U(1)_X$ charge at the fixed points to spontaneously break $U(1)_X$. This can be accomplished with a superpotential term of the form $S\ (\mathbf{1}^+\ \mathbf{1}^- - \Lambda^2)$.

18.4 Proton Decay

Dimension 6 operators for proton decay are suppressed by the inverse power squared of the smallest compactification scale. We will see that this is near the 4D GUT scale and thus the proton lifetime is completely consistent with the experimental bounds. On the other hand, dimension 5 operators for proton decay are only suppressed by the inverse power of the compactification scale. However, if we assume that quarks and leptons only couple to the chiral adjoints containing the Higgs fields, there are no dimension 5 operators for proton decay generated when integrating out the color triplet Higgs fields. This can be attributed to an unbroken $\mathbb{Z}_4^R$ symmetry[7] [353] where the superpotential has charge 2, families have charge 1, $\{\Sigma_{5,6},\ 1^+,\ 1^-\}$ have charge 0, and $\{S,\ \Phi\}$ have charge 2.

18.5 Threshold Corrections

4D SUSY GUTs require extra states to contribute a small amount of threshold corrections at the GUT scale in order to concur with low energy measurements.

[7] We shall consider this $\mathbb{Z}_4^R$ symmetry in more detail in a later chapter.

Conventionally, this quantity of GUT scale threshold corrections (defined at the 4D GUT scale) is defined as:

$$\epsilon_3 = \frac{\alpha_3 - \alpha_{GUT}}{\alpha_{GUT}}. \tag{18.27}$$

The running coupling constants in the 4D MSSM can be summarized by:

$$\alpha_i^{-1}(Q) = \alpha_{GUT}^{-1} + \frac{b_i}{2\pi} log \frac{M_{GUT}}{Q} - \alpha_{GUT}^{-1} \frac{\epsilon_3}{(1+\epsilon_3)} \delta_{i3} \tag{18.28}$$

where δ_{i3} denotes that the term appears only for i=3 (the coupling α_3). The exact amount of threshold corrections required from the extra states is usually model dependent, but they have to be around a few percent level. For the most popular scenarios of MSSM with unified gaugino masses, this number turns out to be about -3%. We would like to calculate the effect of the Kaluza-Klein (KK) tower of infinite states to the running of coupling constants in the orbifold model that we have just constructed. These additional contributions to the running of the coupling constants from KK modes can be written as[8]:

$$\frac{4\pi}{g_i^2(\mu)} = \frac{4\pi}{g^2(\Lambda)} + \sum_{\rho} \Omega_{i,\rho}(\mu) \tag{18.29}$$

where

$$\Omega_{i,\rho}(\mu) \equiv \frac{1}{4\pi} \sum_{(m,n)\in Z} \beta_{i,\rho} \int_{\xi}^{\infty} \frac{dt}{t} e^{-\pi t \frac{M^2_{(m,n),\rho}}{\mu^2}} e^{-\pi \chi t} \tag{18.30}$$

includes one-loop corrections from both massive and massless states in the theory. ξ is the ultraviolet (UV) regulator introduced since the integral is UV-divergent. χ is an infrared (IR) regulator introduced since the above quantity diverges for the special case when there are massless states in the KK tower. The corrections come from each state ρ that appears in the spectrum, with an associated beta-function coefficient, $\beta_{i,\rho}$, summarized in Table 18.2 and mass, $M^2_{(m,n),\rho}$, as calculated in the previous section:

$$M^2_{(m,n),\rho} = \frac{(m + \frac{I_\rho}{4})^2}{R_5^2} + \frac{(n + \frac{I_\rho}{4})^2}{R_6^2} \tag{18.31}$$

We evaluate the expression in Eq. (18.30) in three different regions on the m-n plane shown in Fig. 18.2 and then sum up the contributions to find the total corrections to

[8] We have followed the analysis of [354] in what follows. The details can be found in Appendix 1.

Table 18.2 Nomenclature, quantum numbers, and beta-function coefficients for the various states in the spectrum

Quantum number	Name	Type	b_1	b_2	b_3	Type	b_1	b_2	b_3
$(\mathbf{8},1)_0$	g	C	0	0	3	V	0	0	−9
$(1,\mathbf{3})_0$	w	C	0	2	0	V	0	−6	0
$(\mathbf{3},\mathbf{2})_{\pm 5/3}$	$X, \bar{X}$	C	5/2	3/2	1	V	−15/2	−9/2	−3
$(\mathbf{3},1)_{\pm 2/3}$	$T, \bar{T}$	C	1/5	0	1/2	V	−3/5	0	−3/2
$(1,\mathbf{2})_{\pm 1}$	H_u, H_d	C	3/10	1/2	0	V	−9/10	−3/2	0

the couplings.[9] We will show that for the special case $R_5 = R_6$, we obtain precise gauge coupling unification [PGCU].

18.6 States at m = 0 and n = 0

In this case, the contribution to the threshold corrections is:

$$\Omega^{00}_{i,\rho}(\mu) = \frac{1}{4\pi}\beta_{i,\rho}\int_{\xi}^{\infty}\frac{dt}{t}e^{-\pi t\frac{M^2_{(0,0),\rho}}{\mu^2}}e^{-\pi\chi t} \tag{18.32}$$

We saw earlier that the only states at the m=0, n=0 point are the (++) modes. The (++) modes come from the N=1 SUSY vector fields $g, w, b, X, \bar{X}$, and chiral adjoint fields $T, \bar{T}, H_u, H_d$. The beta-function coefficients for these states are summarized in Table 18.2. Using the results from Appendix, we find:

$$\begin{aligned}\Omega^{00}_i = &\frac{b_i^{++}(I_\rho = 0)}{4\pi}\Gamma\left[0, \pi\xi\chi\right] \\ &+\frac{b_i^{++}(I_\rho = 2)}{4\pi}\Gamma\left[0, \pi\xi\left(\frac{1}{4\mu^2R_5^2}+\frac{1}{4\mu^2R_6^2}+\chi\right)\right]\end{aligned} \tag{18.33}$$

[9] Note, on the orbifold fixed point F_2 the gauge group is only $[SU(3)\times SU(2)\times U(1)_Y]\times U(1)_X$. Thus gauge kinetic operators localized at these fixed points do not need to respect the SU(5) symmetry. However we will assume that the bulk gauge kinetic terms dominate over these localized terms when determining the low energy gauge couplings.

18.7 m Axis, n = 0

Figure 18.2 shows that the (++) and (−−) states live only at even n whereas (+−) and (−+) states live at odd n. The absence of states at certain n has to be accounted for while evaluating the integral. The details of evaluating the odd and even integrals are explicitly presented in Appendix 1 and the result is:

$$
\begin{aligned}
\Omega_i^{m0} &= \frac{b_i^{(++)}(I_\rho = 0)}{4\pi}\mathscr{R}_1^E\left[\xi\nu_1, 0, \frac{\delta_1}{\nu_1}\right] + \frac{b_i^{(++)}(I_\rho = 2)}{4\pi}\mathscr{R}_1^E\left[\xi\nu_1, 1/2, \frac{\delta_1}{\nu_1}\right] \\
&\quad + \frac{b_i^{(+-)}(I_\rho = 0)}{4\pi}\mathscr{R}_1^O\left[\xi\nu_1, 0, \frac{\delta_1}{\nu_1}\right] + \frac{b_i^{(+-)}(I_\rho = 2)}{4\pi}\mathscr{R}_1^O\left[\xi\nu_1, 1/2, \frac{\delta_1}{\nu_1}\right] \\
&\quad + \frac{b_i^{(-+)}(I_\rho = 0)}{4\pi}\mathscr{R}_1^O\left[\xi\nu_1, 0, \frac{\delta_1}{\nu_1}\right] + \frac{b_i^{(-+)}(I_\rho = 2)}{4\pi}\mathscr{R}_1^O\left[\xi\nu_1, 1/2, \frac{\delta_1}{\nu_1}\right] \\
&\quad + \frac{b_i^{(--)}(I_\rho = 0)}{4\pi}\mathscr{R}_1^E\left[\xi\nu_1, 0, \frac{\delta_1}{\nu_1}\right] + \frac{b_i^{(--)}(I_\rho = 2)}{4\pi}\mathscr{R}_1^E\left[\xi\nu_1, 1/2, \frac{\delta_1}{\nu_1}\right] \\
\Omega_i^{m0} &= \left(\frac{b_i^{(++)}(I_\rho = 0)}{4\pi} + \frac{b_i^{(--)}(I_\rho = 0)}{4\pi}\right)\mathscr{R}_1\left[4\xi\nu_1, 0, \frac{\chi}{4\nu_1}\right] \\
&\quad + \left(\frac{b_i^{(+-)}(I_\rho = 0)}{4\pi} + \frac{b_i^{(-+)}(I_\rho = 0)}{4\pi}\right) \\
&\quad \times \left(\mathscr{R}_1\left[4\xi\nu_1, \frac{1}{2}, \frac{\chi}{4\nu_1}\right] + \Gamma\left[0, \pi\xi\left(\nu_1 + \chi\right)\right]\right) \\
&\quad + \left(\frac{b_i^{(+-)}(I_\rho = 2)}{4\pi} + \frac{b_i^{(-+)}(I_\rho = 2)}{4\pi}\right)\Gamma\left[0, \pi\xi\left(\frac{\nu_1}{4} + \frac{\nu_2}{4} + \chi\right)\right]
\end{aligned}
$$

where, $\nu_1 = \frac{1}{\mu^2 R_5^2}$, $\nu_2 = \frac{1}{\mu^2 R_6^2}$, and $\delta_1 = \frac{\rho_2}{\mu^2 R_6^2} + \chi$.

The function $\mathscr{R}_1$ is also defined in Appendix 1. In simplifying the above expression, we have also used the fact that when we have complete N=4 SUSY in 4D, the beta-function coefficients sum up to zero.

$$
b_i^{(++)} + b_i^{(+-)} + b_i^{(-+)} + b_i^{(--)} = 0 \tag{18.34}
$$

for all i.[10]

[10]We have complete N=4 SUSY in 4D when we have one vector multiplet and three chiral multiplets. In terms of the N=1 fields in 4D, the beta-function coefficients are given by:

$$
b_G = 3C_2(G) - N_{\text{chiral}}T(R) \tag{18.35}
$$

18.8 n Axis, m = 0

Along this axis, the calculation is similar to the previous case in the sense that the states exist only at certain n. The (++) and (−+) states live only at even n whereas (+−) and (−−) states live at odd n. Again, using the relations in Appendix 1 and evaluating the integrals, we get:

$$
\begin{aligned}
\Omega_i^{0n} =\ & \frac{b_i^{(++)}(I_\rho=0)}{4\pi}\mathscr{R}_1^E\left[\xi\nu_2, 0, \frac{\delta_2}{\nu_2}\right] + \frac{b_i^{(++)}(I_\rho=2)}{4\pi}\mathscr{R}_1^E\left[\xi\nu_2, 1/2, \frac{\delta_2}{\nu_2}\right] \\
& + \frac{b_i^{(+-)}(I_\rho=0)}{4\pi}\mathscr{R}_1^O\left[\xi\nu_2, 0, \frac{\delta_2}{\nu_2}\right] + \frac{b_i^{(+-)}(I_\rho=2)}{4\pi}\mathscr{R}_1^O\left[\xi\nu_2, 1/2, \frac{\delta_2}{\nu_2}\right] \\
& + \frac{b_i^{(-+)}(I_\rho=0)}{4\pi}\mathscr{R}_1^E\left[\xi\nu_2, 0, \frac{\delta_2}{\nu_2}\right] + \frac{b_i^{(-+)}(I_\rho=2)}{4\pi}\mathscr{R}_1^E\left[\xi\nu_2, 1/2, \frac{\delta_2}{\nu_2}\right] \\
& + \frac{b_i^{(--)}(I_\rho=0)}{4\pi}\mathscr{R}_1^O\left[\xi\nu_2, 0, \frac{\delta_2}{\nu_2}\right] + \frac{b_i^{(--)}(I_\rho=2)}{4\pi}\mathscr{R}_1^O\left[\xi\nu_2, 1/2, \frac{\delta_2}{\nu_2}\right] \\
\Omega_i^{0n} =\ & \left(\frac{b_i^{(++)}(I_\rho=0)}{4\pi} + \frac{b_i^{(-+)}(I_\rho=0)}{4\pi}\right)\mathscr{R}_1\left[4\xi\nu_2, 0, \frac{\chi}{4\nu_2}\right] \\
& + \left(\frac{b_i^{(+-)}(I_\rho=0)}{4\pi} + \frac{b_i^{(--)}(I_\rho=0)}{4\pi}\right) \\
& \times \left(\mathscr{R}_1\left[4\xi\nu_2, \frac{1}{2}, \frac{\chi}{4\nu_2}\right] + \Gamma\left[0, \pi\xi\left(\nu_2+\chi\right)\right]\right) \\
& + \left(\frac{b_i^{(+-)}(I_\rho=2)}{4\pi} + \frac{b_i^{(--)}(I_\rho=2)}{4\pi}\right)\Gamma\left[0, \pi\xi\left(\frac{\nu_2}{4} + \frac{\nu_1}{4} + \chi\right)\right]
\end{aligned}
\tag{18.36}
$$

where, $\nu_1 = \frac{1}{\mu^2 R_5^2}$, $\nu_2 = \frac{1}{\mu^2 R_6^2}$, and $\delta_2 = \frac{\rho_1}{\mu^2 R_5^2} + \chi$ as defined in Appendix 1.

18.9 Off the Axes

This case turns out to be rather simple since all the parity eigenstates live at all (m,n) $\neq$ 0. This includes one vector and three chiral adjoint multiplets for every state and they form complete N=4 supersymmetry. Thus these excited KK modes do not contribute anything to the running of the coupling constants.

18.10 Putting it All Together

The contribution from the four individual cases can be put together with the appropriate beta-function coefficients. In the limit that the regulators can be set to zero, they can be combined with the mass scale μ and replaced by their relevant UV and IR scales.

$$Q^2 \equiv \pi e^{\gamma} \chi \mu^2 \Big|_{\chi \to 0} \qquad \Lambda^2 \equiv \frac{\mu^2}{\xi} \Big|_{\xi \to 0} \tag{18.37}$$

The functions Γ and $\mathscr{R}_1$ in these limits simplify and these simplified expressions are summarized in Appendix 2. The final expression for the threshold corrections at the scale Q coming all the KK states that exist in the system are given by:

$$\begin{aligned}\Omega_i(Q) = & \frac{b_i^{++}(I_\rho = 0)}{4\pi} \ln \frac{\Lambda^2}{Q^2} + \left(\frac{b_i^{+-}(I_\rho = 0) + b_i^{-+}(I_\rho = 0)}{4\pi} \right) \ln \left[\frac{\pi \Lambda}{2M_5} \right]^2 \\ & + \left(\frac{b_i^{+-}(I_\rho = 0) + b_i^{--}(I_\rho = 0)}{4\pi} \right) \ln \left[\frac{\pi \Lambda}{2M_6} \right]^2 \\ & + \frac{b_i^{+-}(I_\rho = 2)}{4\pi} \ln \left[\frac{4\Lambda^2}{M_5^2 + M_6^2} \right]\end{aligned} \tag{18.38}$$

where the scales $M_i, i = 5, 6$ are rescaled compactification scales, i.e. $M_i = \frac{\sqrt{\pi e^{\gamma}}}{R_i}$. Note that to arrive at this result, we have used the spectrum in Fig. 18.2 with mass eigenvalues as shown in Eq. (18.31) (Table 18.3).

One important feature of this expression is that it tells us that there are no power-law corrections to the couplings at any scale. This is unlike generic scenarios of a (4+δ)D model with δ compactified dimensions. The couplings typically receive power-law corrections proportional to $\left(\frac{\Lambda}{M_c}\right)^{\delta}$, where M_c is the smallest compactification scale. Therefore, we should have expected quadratic corrections to the

Table 18.3 Beta-function coefficients relevant for Eq. (18.38)

Coefficients	(b_1, b_2, b_3)
$b_i^{++}(I_\rho = 0)$	$(\frac{33}{5}, 1, -3)$
$b_i^{+-}(I_\rho = 0) + b_i^{-+}(I_\rho = 0)$	$(-\frac{6}{5}, 2, 6)$
$b_i^{+-}(I_\rho = 0) + b_i^{--}(I_\rho = 0)$	$(\frac{6}{5}, 6, 6)$
$b_i^{+-}(I_\rho = 2)$	$(\frac{27}{5}, 3, 3)$

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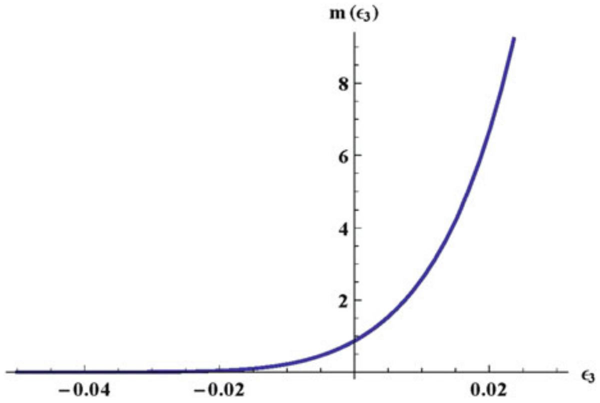

Fig. 18.3 The figure shows the dependence of $m = \left(\frac{M_5}{M_6}\right)^2$ on ϵ_3. The statement that MSSM requires small threshold corrections at the GUT scale translates to anisotropic compactification. Note, for $M_5 = M_6$ we have PGCU. *Reprinted from Nuclear Physics B 868, A. Anandakrishnan and S. Raby, "SU(6) GUT breaking on a projective plane," Page 641, Copyright (2013), with permission from Elsevier*

couplings in the 6D model considered here. It turns out that the quadratic corrections vanish due to the initial N=4 SUSY. This feature was also observed in [346] where an $SU(6)$ theory was studied with N=2 SUSY in 6D. The model discussed in [346], however had an effective 5D limit. Hence there were additional linear corrections to the couplings. In the model discussed here, the compactification takes the 6D theory directly down to 4D and hence we find only logarithmic corrections to the couplings. Moreover, as shown in the next section, the logarithmic corrections are consistent with gauge coupling unification above the scale $\sqrt{M_5^2 + M_6^2}$. Moreover, if we take $M_5 = M_6$ then we obtain precise gauge coupling unification at this compactification scale (Fig. 18.3).

18.11 Results and Discussion

At the lowest compactification scale (largest compactification radius), we have 6D orbifold and the 4D MSSM, respectively:

$$\alpha_i^{-1}(Q) = \alpha^{-1}(\Lambda) + \sum_\rho \Omega_{i,\rho}(Q)$$

$$\alpha_i^{-1}(Q) = \alpha_{GUT}^{-1} + \frac{b_i}{2\pi} log \frac{M_{GUT}}{Q} - \alpha_{GUT}^{-1} \frac{\epsilon_3}{(1+\epsilon_3)} \delta_{i3}$$

We have three sets of equations, one for each coupling of SU(3) $\times$ SU(2) $\times$ U(1)$_Y$ and four unknowns: Λ, M_5, M_6, and $\alpha(\Lambda)$, the unified coupling constant of the

Table 18.4 The table shows a benchmark point for choice 1 and choice 2

	ϵ_3 (%)	M_5	M_6	Λ	$\alpha^{-1}(\Lambda)$
Point 1	−3.0	0.174×10^{16}	2.08×10^{16}	6.0×10^{17}	13.57
Point 2	0.0	3.39×10^{16}	3.64×10^{16}	6.0×10^{17}	17.47
Point 3	+3.0	1.37×10^{17}	3.44×10^{16}	6.0×10^{17}	18.70

We fix α_{GUT}^{-1} to be 24 and M_{GUT} to be 3×10^{16} GeV for both the points. The smallest compactification scale is naturally of the order of the 4D GUT scale. All scales are in GeV. *Reprinted from Nuclear Physics B 868, A. Anandakrishnan and S. Raby, "SU(6) GUT breaking on a projective plane," Page 641, Copyright (2013), with permission from Elsevier*

orbifold theory, given M_{GUT} and ϵ_3 from the 4D MSSM. We find that we can uniquely solve for M_5 and M_6 in terms of M_{GUT} and ϵ_3 and we obtain a curve in the $\alpha - \Lambda$ plane. The details of the solution are elaborated in Appendix 3 and we summarize the solutions obtained:

$$\begin{aligned} M_5 &= \left(m(\epsilon_3)^{(\mathscr{G}-\mathscr{H})/2}(m(\epsilon_3)+1)^{\mathscr{H}/2}e^{\mathscr{I}/2}\right) M_{GUT} \\ M_6 &= \left(m(\epsilon_3)^{(\mathscr{G}-\mathscr{H}-1)/2}(m(\epsilon_3)+1)^{\mathscr{H}/2}e^{\mathscr{I}/2}\right) M_{GUT} \\ \alpha^{-1}(\Lambda) &= -\frac{3}{\pi}\ln\frac{\Lambda^2}{M_{GUT}^2} + \frac{3}{\pi}\ln\left(m(\epsilon_3)^{(\mathscr{G}-\mathscr{H})}(m(\epsilon_3)+1)^{\mathscr{H}}e^{\mathscr{I}}\right) \\ &\quad + \ln\left(m(\epsilon_3)^{(\mathscr{L}-\mathscr{M})}(m(\epsilon_3)+1)^{\mathscr{M}}e^{\mathscr{N}}\right) \end{aligned} \tag{18.39}$$

The coefficients $\mathscr{G}, \mathscr{H}, \mathscr{I}$ and $\mathscr{N}$ are given in Table 18.5 in Appendix 3. To analyze the GUT scale threshold corrections, we fix α_{GUT}^{-1} to be 24 in all further calculations. Benchmark points are shown in Table 18.4. The ratio of M_5 and $M_6 = m$, depends only on ϵ_3 and is shown in Fig. 18.3. The value of m sets the hierarchy between the two compactification scales, M_5 and M_6. We analyzed the particle spectrum at intermediate energies in the cases when (i) $M_5 \ll M_6$ (ii) $M_6 \ll M_5$ and (iii) $M_5 = M_6$ to determine the scale associated with the unification of $SU(3) \times SU(2) \times U(1)_Y$ gauge groups.[11] Also, to determine if the $SU(6)$ was broken down to a subgroup at these intermediate scales, reflecting the two step GUT breaking procedure that we employed. We find two unification scales—the SM gauge group unifies to an $SU(3) \times SU(3) \times U(1)$ at the scale M_5 in all the above three cases. Then further at the scale $\sqrt{M_5^2 + M_6^2}$ there is another unification scale associated with $SU(3) \times SU(3)$ unification to the $SU(6)$ GUT. It is important to note, that even though the gauge symmetry is broken locally at the fixed point F_2, there are no logarithmic corrections to gauge coupling unification above the scale $\sqrt{M_5^2 + M_6^2}$ (see footnote 9). This is due to the effective N = 4 supersymmetry in the bulk.

[11] When $M_6 \ll M_5$ we have KK modes transforming as Higgs doublets and chiral adjoints under the SM gauge group.

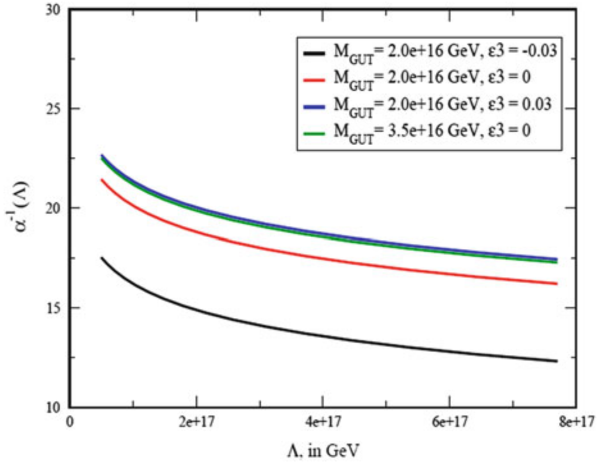

Fig. 18.4 Once M_5 and M_6 are solved for uniquely, we are left with a curve in the $\alpha^{-1} - \Lambda$ plane, as expressed in Eq. (18.39). The unified coupling at the cut-off scale is in the perturbative regime. *Reprinted from Nuclear Physics B 868, A. Anandakrishnan and S. Raby, "SU(6) GUT breaking on a projective plane," Page 641, Copyright (2013), with permission from Elsevier*

It is also interesting to note that the standard scenarios of the MSSM can be embedded in an isotropic or anisotropic orbifold. We find that in the anisotropic as well as isotropic ($M_5 \sim M_6$) cases, the lowest compactification scale is around the 4D GUT scale, making it possible to connect the compactification scale and the 4D GUT scale. For three benchmark points, the curve in the $\alpha^{-1}(\Lambda) - \Lambda$ plane, from Eq. (18.39) is shown in Fig. 18.4. Finally we note that the values of $\alpha^{-1}(\Lambda)$, Λ may be consistent with perturbative heterotic string boundary conditions. In the weakly coupled regime of the heterotic string, the value of the GUT coupling constant at the string scale is given by Dundee and Raby [355]:

$$\alpha^{-1}(\Lambda = M_{string}) = \frac{1}{8}\left(\frac{m_{Pl}}{M_{string}}\right)^2 \tag{18.40}$$

For example, consider the case $M_{GUT} = 2 \times 10^{16}$ GeV and $\epsilon_3 = 0$ in Fig. 18.4. With $\Lambda = 2 \times 10^{17}$ GeV, $m_{Pl} = 2.4 \times 10^{18}$ GeV, we find $\alpha^{-1}(\Lambda) \simeq 18$ which is consistent with Eq. (18.40).

18.12 Summary

In this chapter, we discussed a supersymmetric $SU(6)$ gauge theory on an orbifold with the topology of a real projective plane. The compact space was obtained in two steps by orbifolding a rotation and a freely-acting roto-translation. In the process, the gauge symmetry was broken down from $SU(6)$ to $SU(5) \times U(1)_X$ and the N=4 SUSY was reduced to N=2. To further break the $SU(5)$ down to the Standard Model, we introduced a non-zero Wilson line along the fifth and sixth directions. This helped to eliminate the unwanted light states like the Higgs triplets and to break N=2 to N=1 SUSY.

We calculated the Kaluza Klein spectrum of states coming from this orbifolding and also calculated the threshold corrections coming from these states at the 4D grand unification scale. We find that the threshold corrections coming from the KK states due to compactification on an orbifold with the topology of RP^2 are at the percent level allowing for realistic 4D MSSM. The solutions allow for threshold corrections to be between $\{-3\%, +2\%\}$ allowing for the standard universal gaugino mass scenario like CMSSM or the non-universal gaugino mass scenarios (especially lighter gluinos as discussed in [341, 356]). There have been previous calculations of threshold corrections in orbifold GUT models on various orbifolds with local and non-local GUT breaking. We have already pointed out that unlike in other scenarios we do not get power law running of couplings above the compactification scale due to the large N=2 in 6D. The advantage of not having such large power-law corrections is that we do not lose any predictability due to UV scale physics. We should point out that in the work of Trapletti [344], the author considered a non-local GUT breaking and concluded that the running of couplings stops precisely above the compactification scale. We however find that there are small finite threshold corrections at all scales.

There is one interesting case where the compactification scale, M_c, is the GUT scale, M_{GUT}, i.e. when gauge couplings unify precisely at the compactification scale. This occurs when $M_5 = M_6 = M_c$. At this scale, all KK modes have mass of order M_c. Above this scale $SU(5)$ symmetry is restored and all gauge couplings unify with $\epsilon_3 = 0$, i.e. *we obtain precise gauge coupling unification*.

Our analysis was a bottom-up approach studying the phenomenology of models on an orbifold with the topology of a projective plane. It would be interesting to explore the possibility of embedding these orbifold GUTs into a more fundamental theory, like string theory. On the other hand, it would be equally interesting to study low energy features like SUSY breaking and spectra. Finally, since the compactification scale is naturally around the 4D GUT scale or larger, one does not have to worry about proton decay from dimension 6 operators. Moreover, proton decay from dimension 5 operators vanishes due to a discrete R symmetry.

Appendix 1: Kaluza-Klein Integrals

In order to compute the threshold corrections coming from an infinite tower of Kaluza-Klein states, we would like to evaluate the following integral (see Eq. (18.30)):

$$\sum_{(m,n)\in Z} \int_{\xi}^{\infty} \frac{dt}{t} e^{-\pi t \frac{M^2_{(m,n)}}{\mu^2}} e^{-\pi \chi t} \tag{18.41}$$

where, χ and ξ are IR and UV regulators, and the $M_{(m,n)}$ are the masses of the (m, n)th KK mode. In the presence of Wilson lines they are given by:

$$M^2_{(m,n)} = \frac{(m+\rho_1)^2}{R_5^2} + \frac{(n+\rho_2)^2}{R_6^2} \tag{18.42}$$

where in the scenario that we have, ρ_1 can be either 0 or 2 and ρ_2 is always 0. In general, we can solve the integral following Ghilencea [354] who evaluates the integral for the cases of one and two extra-dimensions. Again, in the current scenario that we have, we find that we only need to evaluate this integral in its one-dimensional limit. We follow Ghilencea and evaluate a 1-dimensional Kaluza-Klein integral of the form:

$$\mathscr{R}_1\,[\xi, \rho, \delta] = \sum_{m\in Z}{}' \int_\xi^\infty \frac{dt}{t} e^{-\pi t\left[(m+\rho)^2+\delta\right]} \tag{18.43}$$

where the prime over the summation in the second term represents that m $\neq$ 0, but runs over all other integer values.

We can make use of the Poisson re-summation formula:

$$\sum_{n\in Z} e^{-\pi A(n+\sigma)^2} = \frac{1}{\sqrt{A}} \sum_{\tilde{n}\in Z} e^{-\pi A^{-1}\tilde{n}^2+2i\pi\tilde{n}\sigma} \tag{18.44}$$

to evaluate this integral. We have,

$$\begin{aligned}
\mathscr{R}_1\,[\xi, \rho, \delta] &= \int_\xi^\infty \frac{dt}{t}\left[-e^{-\pi t\rho^2} + \sum_m e^{-\pi t(m+\rho)^2}\right] e^{-\pi\delta t} \\
&= \int_\xi^\infty \frac{dt}{t}\left[-e^{-\pi t\rho^2} + \frac{1}{\sqrt{t}} + \frac{1}{\sqrt{t}} \sum_m{}' e^{-\pi m^2/t+2i\pi m\rho}\right] e^{-\pi\delta t} \\
&= -\Gamma\left[0, \pi\xi(\delta+\rho^2)\right] + \frac{2e^{-\pi\delta\xi}}{\sqrt{\xi}} + 2\pi\sqrt{\delta}\mathrm{Erf}\left[\sqrt{\pi\delta\xi}\right] \\
&\quad -\log|2\,\sin\pi(\rho+i\sqrt{\delta})|^2
\end{aligned} \tag{18.45}$$

where, it has been assumed that $\xi \ll 1$ while evaluating the integral $\int_\xi^\infty \frac{dt}{t} e^{-\pi m^2/t-\pi\delta t}$ and Ghilencea [354] shows that the error by doing so vanishes when ξ is small.

We summarize the result of this integral in various useful limits:

- (m,n) = (0,0)

$$\int_{\xi}^{\infty} \frac{dt}{t} e^{-\pi t \frac{\rho_1^2}{R_5^2 \mu^2}} e^{-\pi \chi t} = \int_{\xi}^{\infty} \frac{dt}{t} e^{-\pi(\rho+\chi)t}$$
$$= \Gamma\left[0, \pi\xi(\chi+\rho)\right] \tag{18.46}$$

where $\rho = \frac{\rho_1^2}{R_5^2\mu^2} + \frac{\rho_2^2}{R_6^2\mu^2}$

- n=0, m $\neq 0$

$$\sum_{m\in Z}' \int_{\xi}^{\infty} \frac{dt}{t} e^{-\pi t \frac{\frac{(m+\rho_1)^2}{R_5^2} + \frac{\rho_2^2}{R_6^2}}{\mu^2}} e^{-\pi \chi t} = \sum_{m\in Z}' \int_{\xi \nu_1}^{\infty} \frac{dt}{t} e^{-\pi t (m+\rho_1)^2} e^{-\pi \frac{\delta_1}{\nu_1} t}$$
$$= \mathscr{R}_1\left[\xi\nu_1, \rho_1, \frac{\delta_1}{\nu_1}\right] \tag{18.47}$$

where, $\nu_1 = \frac{1}{\mu^2 R_5^2}$ and $\delta_1 = \chi + \frac{\rho_2^2}{\mu^2 R_6^2}$.

- m=0, n $\neq 0$
Similar to the previous case with some parameters interchanged, we have:

$$\sum_{n\in Z}' \int_{\xi}^{\infty} \frac{dt}{t} e^{-\pi t \frac{\frac{\rho_1^2}{R_5^2} + \frac{(n+\rho_2)^2}{R_6^2}}{\mu^2}} e^{-\pi \chi t} = \sum_{n\in Z}' \int_{\xi \nu_2}^{\infty} \frac{dt}{t} e^{-\pi t (n+\rho_2)^2} e^{-\pi \frac{\delta_2}{\nu_2} t}$$
$$= \mathscr{R}_1\left[\xi\nu_2, \rho_2, \frac{\delta_2}{\nu_2}\right] \tag{18.48}$$

where, $\nu_2 = \frac{1}{\mu^2 R_6^2}$ and $\delta_2 = \chi + \frac{\rho_1^2}{\mu^2 R_5^2}$

- Since the spectrum we are interested in has states that live either at odd or even integers, it is write down the result of this integral in these limit that the summation is over either even or odd integers:
$n = 0, m \neq 0; m = even$

$$\mathscr{R}_1^E\left[\xi\nu_1, \rho_1, \frac{\delta_1}{\nu_1}\right] = \sum_{m\in Z}' \int_{\xi}^{\infty} \frac{dt}{t} e^{-\pi t \frac{\frac{(2m+\rho_1)^2}{R_5^2} + \frac{\rho_2^2}{R_6^2}}{\mu^2}} e^{-\pi \chi t}$$
$$= \sum_{m\in Z}' \int_{\xi}^{\infty} \frac{dt}{t} e^{-4\pi t \frac{\frac{(m+\frac{\rho_1}{2})^2}{R_5^2} + \frac{\rho_2^2}{4R_6^2}}{\mu^2}} e^{-\pi \chi t/4}$$

$$= \sum_{m \in Z}' \int_{4\xi\nu_1}^{\infty} \frac{dt}{t} e^{-\pi t (m+\rho_1)^2} e^{-\pi \frac{\delta_1}{4\nu_1} t}$$
$$= \mathscr{R}_1 \left[4\xi\nu_1, \frac{\rho_1}{2}, \frac{\delta_1}{4\nu_1} \right] \tag{18.49}$$

where, $\nu_1 = \frac{1}{\mu^2 R_5^2}$ and $\delta_1 = \chi + \frac{\rho_2^2}{\mu^2 R_6^2}$ is the same as previously defined.
$n = 0, m \neq 0$; $m = odd$

$$\mathscr{R}_1^O \left[\xi\nu_1, \rho_1, \frac{\delta_1}{\nu_1} \right] = \sum_{m \in Z} \int_{\xi}^{\infty} \frac{dt}{t} e^{-\pi t \frac{\frac{(2m-1+\rho_1)^2}{R_5^2} + \frac{\rho_2^2}{R_6^2}}{\mu^2}} e^{-\pi \chi t}$$
$$= \sum_{m \in Z} \int_{\xi}^{\infty} \frac{dt}{t} e^{-4\pi t \frac{\frac{(m+\frac{\rho_1 - 1}{2})^2}{R_5^2} + \frac{\rho_2^2}{4R_6^2}}{\mu^2}} e^{-\pi \chi t/4}$$
$$= \int_{\xi\nu_1}^{\infty} \frac{dt}{t} e^{-\pi t \frac{(\rho_1 - 1)^2}{4}} e^{-\pi \frac{\delta_1}{4\nu_1} t}$$
$$+ \sum_{m \in Z}' \int_{4\xi\nu_1}^{\infty} \frac{dt}{t} e^{-\pi t (m + \frac{\rho_1 - 1}{2})^2} e^{-\pi \frac{\delta_1}{4\nu_1} t}$$
$$= \Gamma\left[0, \pi\xi(\nu_1(\rho_1 - 1)^2 + \delta_1)\right] + \mathscr{R}_1 \left[4\xi\nu_1, \frac{\rho_1 - 1}{2}, \frac{\delta_1}{4\nu_1} \right] \tag{18.50}$$

In order to write the result of the integral in terms of the original $\mathscr{R}_1$, we separate the zeroth term from the rest in the summation. It is also useful to note that the function $\mathscr{R}_1$ is even in ρ and hence:

$$\mathscr{R}_1 [\xi, \rho, \delta] = \mathscr{R}_1 [\xi, -\rho, \delta] \tag{18.51}$$

Appendix 2: Useful Limits of Relevant Functions

The result of the Kaluza-Klein integrals were evaluated in the previous section, in terms of the two functions, $\Gamma[0, \pi\xi\chi]$ and $\mathscr{R}_1 [\xi, \rho, \delta]$. χ and ξ are the regulators and in the limit that they are zero, we can replace them with the relevant mass scales.

$$Q^2 \equiv \pi e^{\gamma} \chi \mu^2 \Big|_{\chi \to 0} \qquad \Lambda^2 \equiv \frac{\mu^2}{\xi} \Big|_{\xi \to 0} \tag{18.52}$$

As evaluated in the previous section:

$$\mathscr{R}_1\,[\xi,\rho,\delta] = -\Gamma\left[0,\pi\xi(\delta+\rho^2)\right] + \frac{2e^{-\pi\delta\xi}}{\sqrt{\xi}} + 2\pi\sqrt{\delta}\mathrm{Erf}\left[\sqrt{\pi\delta\xi}\right]$$
$$-\log|2\,\sin\pi(\rho+i\sqrt{\delta})|^2$$

We use the following expansions:

$$-\,\Gamma\,[0,z] = \gamma + \ln z + \sum_{k\geq 1}\frac{(-z)^k}{k!\,k} \qquad z>0 \tag{18.53}$$

$$\mathrm{Erf}\,[x] = \frac{2x}{\sqrt{\pi}} - \frac{2x^3}{3\sqrt{\pi}} + \mathscr{O}(x^5) \qquad x \ll 1 \tag{18.54}$$

Then,

$$\begin{aligned}\Gamma\,[0,\pi\xi\chi] &= -\gamma - \ln\pi\xi\chi\\ &= -\ln\pi\frac{e^{\gamma}\mu^2 Q^2}{\Lambda^2\pi e^{\gamma}\mu^2}\\ &= -\ln\frac{Q^2}{\Lambda^2}\end{aligned} \tag{18.55}$$

With these approximations, $\mathscr{R}_1\,[\xi,\rho,\delta]$ simplifies to:

$$\mathscr{R}_1\,[\xi,\rho,\delta] = -\ln\left[4\pi e^{-\gamma}\frac{1}{\xi}e^{-2/\sqrt{\xi}}\right] - \ln\left|\frac{\sin(\rho+i\sqrt{\delta})}{\pi(\rho+i\sqrt{\delta})}\right|^2 \tag{18.56}$$

We summarize, the various terms that come up in the calculation of threshold corrections in Sect. 18.5. In the expressions below, we have also introduced the compactifications scales $M_5 = \sqrt{\pi e^{\gamma}}/R_5$ and $M_6 = \sqrt{\pi e^{\gamma}}/R_6$.

$$\begin{aligned}\Gamma\,[0,\pi\xi\chi] &= \ln\frac{\Lambda^2}{Q^2}\\ \Gamma\,[0,\pi\xi\nu_1] &= -\gamma - \ln\frac{\pi}{\Lambda^2 R_5^2}\\ &= -\ln\left[\frac{M_5^2}{\Lambda^2}\right]\\ \Gamma\,[0,\pi\xi\nu_2] &= -\gamma - \ln\frac{\pi}{\Lambda^2 R_6^2}\\ &= -\ln\left[\frac{M_6^2}{\Lambda^2}\right]\end{aligned}$$

$$
\begin{aligned}
\mathscr{R}_1\left[4\xi\nu_1, 0, \frac{\chi}{4\nu_1}\right] &= -\ln\left[\pi e^{-\gamma-\Lambda R_5}(\Lambda R_5)^2\right] \\
\mathscr{R}_1\left[4\xi\nu_1, \frac{1}{2}, \frac{\chi}{4\nu_1}\right] &= -\ln\left[\pi e^{-\gamma-\Lambda R_5}(\Lambda R_5)^2\right] - \ln\left[\frac{2}{\pi}\right]^2 \\
\mathscr{R}_1\left[4\xi\nu_2, 0, \frac{\chi}{4\nu_2}\right] &= -\ln\left[\pi e^{-\gamma-\Lambda R_6}(\Lambda R_6)^2\right] \\
\mathscr{R}_1\left[4\xi\nu_2, \frac{1}{2}, \frac{\chi}{4\nu_2}\right] &= -\ln\left[\pi e^{-\gamma-\Lambda R_6}(\Lambda R_6)^2\right] - \ln\left[\frac{2}{\pi}\right]^2 \\
\Gamma\left[0, \pi\xi\left(\frac{\nu_1}{4} + \frac{\nu_2}{4}\right)\right] &= -\ln\left[\frac{M_5^2 + M_6^2}{4\Lambda^2}\right]
\end{aligned}
\tag{18.57}
$$

Appendix 3: 6D → 4D Matching

We calculated the corrections to the gauge couplings coming from the KK states of the 6D orbifold model that was constructed. At the lowest compactification scale (largest compactification radius), we said that the couplings from 4D MSSM and 6D orbifold model should match. In this section, we will compare the two sets of equations, from the two theories:

$$
\begin{aligned}
\alpha_i^{-1}(Q) &= \alpha^{-1}(\Lambda) + \sum_\rho \Omega_{i,\rho}(Q) \\
\alpha_i^{-1}(Q) &= \alpha_{GUT}^{-1} + \frac{b_i}{2\pi} log \frac{M_{GUT}}{Q} - \alpha_{GUT}^{-1}\frac{\epsilon_3}{(1+\epsilon_3)}\delta_{i3}
\end{aligned}
$$

and solve for the three scales of the orbifold model, Λ, M_5, and M_6 as well as coupling constant, α at the cut-off scale.

Since the two expressions have to match at all scales below the smallest compactification scale of the orbifold model, we can rewrite the above two equations as:

$$
\begin{aligned}
&\alpha_{GUT}^{-1} + \frac{b_i}{4\pi}\ln\frac{M_{GUT}^2}{Q^2} - \alpha_{GUT}^{-1}\frac{\epsilon_3}{(1+\epsilon_3)}\delta_{i3} \\
&\quad = \alpha^{-1}(\Lambda) + \frac{b_i^{++}(I_\rho = 0)}{4\pi}\ln\frac{\Lambda^2}{Q^2} + \left(\frac{b_i^{+-}(I_\rho = 0) + b_i^{-+}(I_\rho = 0)}{4\pi}\right)\ln\left[\frac{\pi\Lambda}{2M_5}\right]^2 \\
&\qquad + \left(\frac{b_i^{+-}(I_\rho = 0) + b_i^{--}(I_\rho = 0)}{4\pi}\right)\ln\left[\frac{\pi\Lambda}{2M_6}\right]^2 + \frac{b_i^{+-}(I_\rho = 2)}{4\pi}\ln\left[\frac{4\Lambda^2}{M_5^2 + M_6^2}\right]
\end{aligned}
\tag{18.58}
$$

where we have used the complete expression we estimated for the corrections to couplings in (18.38).

We use the following redefinitions:

$$\frac{b_i^{MSSM}}{4\pi} = \frac{b_i^{++}(I_\rho = 0)}{4\pi} = \beta_i$$
$$\frac{b_i^{+-}(I_\rho = 0) + b_i^{-+}(I_\rho = 0)}{4\pi} = -A_i$$
$$\frac{b_i^{+-}(I_\rho = 0) + b_i^{--}(I_\rho = 0)}{4\pi} = -B_i$$
$$\frac{b_i^{+-}(I_\rho = 2)}{4\pi} = -C_i \tag{18.59}$$

and

$$(A_i + B_i)\ln\left[\frac{\pi}{2}\right]^2 + C_i\ln[4] = D_i \tag{18.60}$$

and hence end up with a set of three equations that can be simply written as:

$$\alpha_{GUT}^{-1} - \alpha^{-1}(\Lambda) - \alpha_{GUT}^{-1}\frac{\epsilon_3}{(1+\epsilon_3)}\delta_{i3} - \beta_i\ln\frac{\Lambda^2}{M_{GUT}^2}$$
$$+A_i\ln\frac{\Lambda^2}{M_5^2} + B_i\ln\frac{\Lambda^2}{M_6^2} + C_i\ln\frac{\Lambda^2}{M_5^2 + M_6^2} + D_i = 0 \tag{18.61}$$

where, $A_i = A_{1i} + A_{2i}$ and $i = 1, 2, 3$. We look at the equations corresponding (i) (i = 1) - (i =2) (ii) i = 2 (iii) i = 3 and solve for Λ, M_5, and M_6. It is usually considered that the 4D unification scale is around 3.0×10^{16} GeV and the couplings at this scale are unified at $\alpha_{GUT}^{-1} = 24$. In standard scenarios of MSSM with gaugino mass unification, $\epsilon_3 = -3\%$. Depending of the spectrum of low energy SUSY, these quantities are subject to change. The first equation we get by simplifying Eq. (18.61) for (i = 1) - (i =2) is:

$$-(\beta_1 - \beta_2)\ln\frac{\Lambda^2}{M_{GUT}^2} + (A_1 - A_2)\ln\frac{\Lambda^2}{M_5^2} + (B_1 - B_2)\ln\frac{\Lambda^2}{M_6^2}$$
$$+(C_1 - C_2)\ln\frac{\Lambda^2}{M_5^2 + M_6^2} + (D_1 - D_2) = 0 \tag{18.62}$$

Defining, $\frac{\Lambda^2}{M_5^2} = X$ and $\frac{\Lambda^2}{M_6^2} = Y$, we get:

$$\ln \frac{\Lambda^2}{M_{GUT}^2} = \left(\frac{A_1 - A_2}{\beta_1 - \beta_2}\right) \ln X + \left(\frac{B_1 - B_2}{\beta_1 - \beta_2}\right) \ln Y$$
$$- \left(\frac{C_1 - C_2}{\beta_1 - \beta_2}\right) \ln \left(\frac{1}{X} + \frac{1}{Y}\right) + \left(\frac{D_1 - D_2}{\beta_1 - \beta_2}\right) \tag{18.63}$$

Next, we look at Eq. (18.61) when $i = 2$:

$$\alpha_{GUT}^{-1} - \alpha^{-1}(\Lambda) - \beta_2 \ln \frac{\Lambda^2}{M_{GUT}^2}$$
$$+ A_2 \ln X + B_2 \ln Y - C_2 \ln \left(\frac{1}{X} + \frac{1}{Y}\right) + D_2 = 0 \tag{18.64}$$

Then, using the expression we just derived in Eq. (18.63), we get an expression for $\alpha^{-1}(\Lambda)$:

$$\alpha^{-1}(\Lambda) = \alpha_{GUT}^{-1} + \left[A_2 - \beta_2 \left(\frac{A_1 - A_2}{\beta_1 - \beta_2}\right)\right] \ln X + \left[B_2 - \beta_2 \left(\frac{B_1 - B_2}{\beta_1 - \beta_2}\right)\right] \ln Y$$
$$- \left[C_2 - \beta_2 \left(\frac{C_1 - C_2}{\beta_1 - \beta_2}\right)\right] \ln \left(\frac{1}{X} + \frac{1}{Y}\right) + \left[D_2 - \beta_2 \left(\frac{D_1 - D_2}{\beta_1 - \beta_2}\right)\right] \tag{18.65}$$

Finally, we look at Eq. (18.61) when $i = 3$, and simplify it using the relations obtained in Eqs. (18.63) and (18.65) and we get a final expression:

$$\left[A_3 - A_2 + \left(\frac{A_1 - A_2}{\beta_1 - \beta_2}\right)(\beta_2 - \beta_3)\right] \ln X + \left[B_3 - B_2 + \left(\frac{B_1 - B_2}{\beta_1 - \beta_2}\right)(\beta_2 - \beta_3)\right] \ln Y$$
$$- \left[C_3 - C_2 + \left(\frac{C_1 - C_2}{\beta_1 - \beta_2}\right)(\beta_2 - \beta_3)\right] \ln \left(\frac{1}{X} + \frac{1}{Y}\right)$$
$$+ \left[D_3 - D_2 + \left(\frac{D_1 - D_2}{\beta_1 - \beta_2}\right)(\beta_2 - \beta_3)\right] - \alpha_{GUT}^{-1} \frac{\epsilon_3}{1 + \epsilon_3} = 0 \tag{18.66}$$

The above three equations can be rewritten in a simple manner as (in the order Eqs. (18.66), (18.63), (18.65)):

$$\begin{aligned}
\mathscr{A}\ln X + \mathscr{B}\ln Y - \mathscr{C}\ln\left(\frac{1}{X}+\frac{1}{Y}\right) + \mathscr{D} &= 0\\
\mathscr{F}\ln X + \mathscr{G}\ln Y - \mathscr{H}\ln\left(\frac{1}{X}+\frac{1}{Y}\right) + \mathscr{I} &= \ln\frac{\Lambda^2}{M_{GUT}^2}\\
\mathscr{K}\ln X + \mathscr{L}\ln Y - \mathscr{M}\ln\left(\frac{1}{X}+\frac{1}{Y}\right) + \mathscr{N} &= \alpha^{-1}(\Lambda)
\end{aligned} \tag{18.67}$$

with,

$$\begin{aligned}
\mathscr{A} &= A_3 - A_2 + (A_1 - A_2)\left(\frac{\beta_2-\beta_3}{\beta_1-\beta_2}\right),\\
\mathscr{B} &= B_3 - B_2 + (B_1 - B_2)\left(\frac{\beta_2-\beta_3}{\beta_1-\beta_2}\right),\\
\mathscr{C} &= C_3 - C_2 + (C_1 - C_2)\left(\frac{\beta_2-\beta_3}{\beta_1-\beta_2}\right),\\
\mathscr{D} &= D_3 - D_2 + (D_1 - D_2)\left(\frac{\beta_2-\beta_3}{\beta_1-\beta_2}\right) - \alpha_{GUT}^{-1}\frac{\epsilon_3}{(1+\epsilon_3)}\\
\mathscr{F} &= \frac{A_1-A_2}{\beta_1-\beta_2},\\
\mathscr{G} &= \frac{B_1-B_2}{\beta_1-\beta_2},\\
\mathscr{H} &= \frac{C_1-C_2}{\beta_1-\beta_2},\\
\mathscr{I} &= \frac{D_1-D_2}{\beta_1-\beta_2},\\
\mathscr{K} &= A_2 - \beta_2\left(\frac{A_1-A_2}{\beta_1-\beta_2}\right),\\
\mathscr{L} &= B_2 - \beta_2\left(\frac{B_1-B_2}{\beta_1-\beta_2}\right),\\
\mathscr{M} &= C_2 - \beta_2\left(\frac{C_1-C_2}{\beta_1-\beta_2}\right),\\
\mathscr{N} &= D_2 - \beta_2\left(\frac{D_1-D_2}{\beta_1-\beta_2}\right) + \alpha_{GUT}^{-1},
\end{aligned}$$

These quantities can be calculated using the beta-function coefficients given in Table 18.3. The numerical values of all the above coefficients are summarized in Table 18.5.

With these coefficients, we get a simple quadratic equation in terms in of the variables X and Y:

$$\left(\frac{Y}{X}\right)^2 + \frac{Y}{X} - \mathrm{Exp}\left(-\frac{7\pi\mathscr{D}(\epsilon_3)}{3}\right) = 0 \tag{18.68}$$

Recall that $X = \frac{\Lambda^2}{M_5^2}$ and $Y = \frac{\Lambda^2}{M_6^2}$, which implies that the above equation turns into a quadratic equation in $\left(\frac{M_5}{M_6}\right)^2$ with the solution.

$$M_5^2 = \frac{-1 \pm \sqrt{1 + 4\mathrm{Exp}\left(-\frac{7\pi\mathscr{D}(\epsilon_3)}{3}\right)}}{2} M_6^2 \tag{18.69}$$

which we write as $M_5 = \sqrt{m(\epsilon_3)}M_6$. The slope m, is the positive solution from the above expression and is shown in Fig. 18.3. The other two equations then yield us M_5 and M_6 uniquely and one expression relating $\alpha^{-1}(\Lambda)$ and Λ.

$$\begin{aligned}
M_5 &= \left(m(\epsilon_3)^{(\mathscr{G}-\mathscr{H})/2}(m(\epsilon_3)+1)^{\mathscr{H}/2}e^{\mathscr{I}/2}\right) M_{GUT} \\
M_6 &= \left(m(\epsilon_3)^{(\mathscr{G}-\mathscr{H}-1)/2}(m(\epsilon_3)+1)^{\mathscr{H}/2}e^{\mathscr{I}/2}\right) M_{GUT} \\
\alpha^{-1}(\Lambda) &= -\frac{3}{\pi}\ln\frac{\Lambda^2}{M_{GUT}^2} + \frac{3}{\pi}\ln\left(m(\epsilon_3)^{(\mathscr{G}-\mathscr{H})}(m(\epsilon_3)+1)^{\mathscr{H}}e^{\mathscr{I}}\right) \\
&\quad +\ln\left(m(\epsilon_3)^{(\mathscr{L}-\mathscr{M})}(m(\epsilon_3)+1)^{\mathscr{M}}e^{\mathscr{N}}\right)
\end{aligned} \tag{18.70}$$

Table 18.5 The coefficients in the expression Eq. (18.67)

Coefficient	Value
$\mathscr{A}$	$-\frac{3}{7\pi}$
$\mathscr{B}$	$\frac{6}{7\pi}$
$\mathscr{C}$	$-\frac{3}{7\pi}$
$\mathscr{D}(\epsilon_3)$	$-\alpha_{GUT}^{-1}\frac{\epsilon_3}{1+\epsilon_3} - \frac{6}{7\pi}\ln\frac{4}{\pi}$
$\mathscr{F}$	$\frac{4}{7}$
$\mathscr{G}$	$\frac{6}{7}$
$\mathscr{H}$	$-\frac{3}{7}$
$\mathscr{I}$	$-\frac{3}{7}\ln 4 - \frac{20}{7}\ln\frac{2}{\pi}$
$\mathscr{K}$	$-\frac{9}{14\pi}$
$\mathscr{L}$	$-\frac{12}{7\pi}$
$\mathscr{M}$	$-\frac{9}{14\pi}$
$\mathscr{N}$	$\alpha_{GUT}^{-1} - \frac{6}{7\pi}\ln 2 - \frac{33}{7\pi}\ln\frac{2}{\pi}$

Chapter 19
Discrete *R* Symmetries for the MSSM and Its Singlet Extensions

In this book we have studied supersymmetric extensions of the Standard Model, including the MSSM, and supersymmetric orbifold GUTs in 4 and higher dimensions. Although they provide a framework for solving the hierarchy problem, they introduce additional problems associated with the μ term, flavor problems and new effective operators which violate baryon and lepton number, leading to proton decay. We have discussed particular mechanisms which can ameliorate each of these problems. The μ problem can be solved by incorporating a symmetry which forbids the μ term at the tree level, but is broken spontaneously by a SUSY breaking VEV, i.e. the so-called Giudice-Masiero mechanism [117], or by a Peccei-Quinn symmetry breaking VEV which results in a μ term and an invisible QCD axion, the so-called Kim-Nilles mechanism [118]. Minimal flavor violation and/or heavy scalars can protect against large flavor violation. This, of course, depends on the SUSY breaking mechanisms.

With regards to proton decay, dangerous dimension four operators can be forbidden by *R*- or matter parity [6, 29, 87], which is an anomaly free $\mathbb{Z}_2$ subgroup of the continuous baryon minus lepton symmetry $U(1)_{B-L}$. Dimension five proton decay operators can be forbidden by 'baryon triality' [89], which combines with matter parity to give 'proton hexality' [90, 357]. The latter is the unique anomaly free discrete non-*R* symmetry forbidding the dangerous operators while allowing the usual Yukawa couplings, the μ-term and the effective neutrino mass operator.[1] However, there are two unpleasant properties of these traditional discrete symmetries. First, they do not allow to address the μ problem. Second, they do not commute with the symmetries of the grand unified theories (GUTs) SU(5) or SO(10) [361].

In [362] a discrete *R* symmetry was identified which can address the μ problem and commutes with SO(10). This $\mathbb{Z}_4^R$ symmetry is anomaly free through cancellation by the Green–Schwarz [GS] mechanism. In [353] it was shown that this $\mathbb{Z}_4^R$ is

[1] Anomaly freedom is believed to be a necessary property of discrete symmetries as otherwise quantum gravity effects may render them inefficient [89, 358–360].

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the unique possibility which commutes with SO(10), and it was pointed out that it also solves the problem associated with dimension five proton decay operators. Furthermore it contains matter parity as a $\mathbb{Z}_2$ subgroup that is left unbroken after supersymmetry breaking. It has been shown that the $\mathbb{Z}_4^R$ symmetry is only allowed in orbifold GUTs [363]. In addition, in orbifold GUTs, a μ term and dimension 5 operators can also be forbidden by a continuous R-symmetry [314]. Finally, dimension 6 operators are very model dependent. They can be enhanced if quarks and leptons reside on a GUT brane with a low energy compactification scale. Or they can be suppressed by placing quarks and leptons in the bulk.

Here we consider, in general, the possible discrete symmetries of the MSSM which commute with SU(5). As we shall see, there are only five possibilities with the simplest one being the $\mathbb{Z}_4^R$. Our analysis applies to singlet extensions of the MSSM as well. We feel that these discrete R symmetries are useful for defining phenomenologically safe low energy theories. Moreover, in the next few chapters we shall discuss the embedding of orbifold GUTs in string theories. In such theories, model dependence is very constrained. Model dependence is essentially rephrased as a choice of the geometry of the extra dimensions. The particle states and their interactions are output once the geometry is chosen. We shall see that discrete symmetries in string theory can be useful to define the low energy theory we call the MSSM.

The chapter is organized as follows. In Sect. 19.1 we prove that there are only five (generation independent) discrete $\mathbb{Z}_M^R$ symmetries which (1) commute with SU(5), (2) allow the usual Yukawa couplings and dimension five neutrino mass operator and (3) address the μ and proton decay problems of the MSSM. Section 19.2 is dedicated to a more detailed discussion of the simplest such symmetry, $\mathbb{Z}_4^R$. In Sect. 19.3 we discuss discrete R symmetries in singlet extensions of the MSSM. In a theory with the usual NMSSM couplings the discrete R symmetries can, apart from suppressing the proton decay rate, provide us with a solution to the NMSSM hierarchy problem. In a different singlet extension, in which the singlet couples quadratically to the Higgs bilinear, we will identify a unique discrete R symmetry capable of solving the μ and strong CP problems simultaneously. Finally, Sect. 19.4 contains our conclusions. In two appendices we present a re-derivation of discrete anomalies in the path integral approach and collect anomaly coefficients for discrete R and non-R symmetries. This chapter is based on the results of [353, 364]. See also [365–367].

19.1 Discrete Symmetries of the MSSM

In this section we discuss discrete symmetries of the MSSM which commute with SU(5) and can solve the μ problem. As we shall see, the assumption that matter $\mathbb{Z}_M$ charges commute with SU(5) allows us to restrict possible $\mathbb{Z}_M$ symmetries of the MSSM, as well as singlet extensions, to only few possibilities. We start in Sect. 19.1

by showing that one cannot address the μ problem with non-R symmetries. In Sect. 19.1.1 we then turn to the discussion of discrete R symmetries, for which we prove that the order M has to divide 24. Finally, in Section "Classification", we classify all possible charge assignments.

Non-R Discrete Symmetries

We start by discussing non-R symmetries. We show that such discrete symmetries that are consistent with SO(10) or SU(5) relations for matter, i.e. universal charges for quarks and leptons, cannot forbid the μ term (cf. the similar discussion in [368]).

Consider a $\mathbb{Z}_M$ symmetry under which the three generations of Q, $\overline{U}$ and $\overline{E}$ carry discrete charge $q^g_{\mathbf{10}}$ while L and $\overline{D}$ carry $q^g_{\overline{\mathbf{5}}}$, where g labels the generation index. Our conventions are given in Section "$\mathbb{Z}_{\boldsymbol{M}}$ and $\mathbb{Z}^{\boldsymbol{R}}_{\boldsymbol{M}}$ Anomaly Coefficients" in Appendix. If the $\mathbb{Z}_M$ charges obey the even stronger SO(10) relations (i.e. $q^g_{\mathbf{10}} = q^g_{\overline{\mathbf{5}}}$), the following discussion applies as well. The anomaly coefficients $A_3 := A_{\mathrm{SU}(3)_C-\mathrm{SU}(3)_C-\mathbb{Z}_M}$, $A_2 := A_{\mathrm{SU}(2)_\mathrm{L}-\mathrm{SU}(2)_\mathrm{L}-\mathbb{Z}_M}$, $A_1 := A_{U(1)_Y-U(1)_Y-\mathbb{Z}_M}$ and $A_0 := A_{\mathrm{grav}-\mathrm{grav}-\mathbb{Z}_M}$ are (cf. Eq. (19.79) in Section "$\mathbb{Z}_{\boldsymbol{M}}$ and $\mathbb{Z}^{\boldsymbol{R}}_{\boldsymbol{M}}$ Anomaly Coefficients" in Appendix)

$$A_3 = \frac{1}{2}\sum_{g=1}^{3}(3\cdot q^g_{\mathbf{10}} + q^g_{\overline{\mathbf{5}}}) \ , \tag{19.1a}$$

$$A_2 = \frac{1}{2}\sum_{g=1}^{3}(3\cdot q^g_{\mathbf{10}} + q^g_{\overline{\mathbf{5}}}) + \frac{1}{2}\,(q_{H_u} + q_{H_d}) \ , \tag{19.1b}$$

$$A_1 = \frac{1}{2}\sum_{g=1}^{3}\left(3\cdot q^g_{\mathbf{10}} + q^g_{\overline{\mathbf{5}}}\right) + \frac{3}{5}\cdot\frac{1}{2}\cdot(q_{H_u} + q_{H_d}) \ , \tag{19.1c}$$

$$A_0 = \sum_{g=1}^{3}\left(10\cdot q^g_{\mathbf{10}} + 5\cdot q^g_{\overline{\mathbf{5}}}\right) + 2q_{H_u} + 2q_{H_d} \ , \tag{19.1d}$$

where the sum runs over the generation indices g and q_{H_u} and q_{H_d} denote the $\mathbb{Z}_M$ charges of the up-type and down-type Higgs doublets, respectively. Anomaly freedom requires

$$(A_{1\leq i\leq 3} \mod \eta) \ = \ \frac{1}{24}(A_0 \mod \eta) \ = \ \rho \tag{19.2}$$

with $\rho \neq 0$ in the case of GS anomaly cancellation (cf. Eq. (19.62) in Section "Discrete Green–Schwarz Mechanism" in Appendix). Here we define

$$\eta := \begin{cases} M & \text{for } M \text{ odd} , \\ M/2 & \text{for } M \text{ even} . \end{cases} \tag{19.3}$$

Condition (19.2) implies

$$A_2 - A_3 = 0 \mod \eta \tag{19.4}$$

and hence, also in the case of generation-dependent $\mathbb{Z}_M$ charges,

$$\frac{1}{2}\left(q_{H_u} + q_{H_d}\right) = 0 \mod \eta . \tag{19.5}$$

On the other hand, the condition that the μ term is allowed is

$$q_{H_u} + q_{H_d} = 0 \mod M . \tag{19.6}$$

We therefore see that, if we demand SU(5) relations for matter charges, a non-R $\mathbb{Z}_M$ symmetry cannot be used to address the μ problem, even if we allow for GS cancellation of anomalies.

19.1.1 Discrete R-symmetries

Having seen that non-R symmetries cannot be used to address the μ problem, we turn to discuss discrete R symmetries. In this subsection, we derive constraints on the order M of $\mathbb{Z}_M^R$ symmetries that can solve the μ problem and accommodate the structure of the MSSM.

A Constraint on the Order M

After adding the contribution of the gauginos and gravitino the anomaly coefficients are

$$A_3^R = \frac{1}{2}\sum_{g=1}^{3}\left(3q_{\mathbf{10}}^g + q_{\overline{\mathbf{5}}}^g\right) - 3 , \tag{19.7a}$$

$$A_2^R = \frac{1}{2}\sum_{g=1}^{3}\left(3q_{\mathbf{10}}^g + q_{\overline{\mathbf{5}}}^g\right) + \frac{1}{2}\left(q_{H_u} + q_{H_d}\right) - 5 , \tag{19.7b}$$

$$A_1^R = \frac{1}{2}\sum_{g=1}^{3}\left(3q_{\mathbf{10}}^g + q_{\overline{\mathbf{5}}}^g\right) + \frac{3}{5}\left[\frac{1}{2}\left(q_{H_u} + q_{H_d}\right) - 11\right] , \tag{19.7c}$$

$$A_0^R = -21 + 8 + 3 + 1 + \sum_{g=1}^{3}\left[10\left(q_{\mathbf{10}}^g - 1\right) + 5\left(q_{\overline{\mathbf{5}}}^g - 1\right)\right]$$
$$+2\left(q_{H_u} + q_{H_d} - 2\right) , \tag{19.7d}$$

where $q_{\mathbf{10}}$, $q_{\overline{\mathbf{5}}}$, q_{H_u} and q_{H_d} denote the R charges of the matter and Higgs superfields, i.e. matter fermions and Higgsinos have charges $q - 1$.

In the case $\rho \neq 0$, the GS mechanism requires the presence of an axion, such that A_0^R is to be amended by the axino/dilatino contribution ($q_{\tilde{a}} = -1$).

Subtracting the coefficients from each other leads to the universality conditions

$$A_2^R - A_3^R = 0 \mod \eta \quad \curvearrowright \quad q_{H_u} + q_{H_d} = 4 \mod 2\eta , \tag{19.8a}$$

$$A_1^R - A_3^R = 0 \mod \eta$$
$$\curvearrowright \quad \frac{3}{5}\left[\frac{1}{2}\left(q_{H_u} + q_{H_d}\right) - 6\right] = 0 \mod \eta . \tag{19.8b}$$

Equation (19.8a) is equivalent to

$$\frac{1}{2}\left(q_{H_u} + q_{H_d}\right) = 2 + \eta\,\ell \tag{19.9}$$

with an integer ℓ. Inserting this into Eq. (19.8b) yields

$$\frac{3}{5}\left[\ell\,\eta - 4\right] = k\,\eta \tag{19.10}$$

with another integer k. Altogether we find

$$[3\,\ell - 5\,k] = 12/\eta = \begin{cases} 24/M , \text{ for } M \text{ even} , \\ 12/M , \text{ for } M \text{ odd} . \end{cases} \tag{19.11}$$

In both cases $24/M$ has to be integer, i.e. M has to divide 24. Thus the possible values of M are $3, 4, 6, 8, 12$ and 24.[2] In what follows, we consider all these possibilities.

[2]We exclude the case $M = 2$ since there are no meaningful order 2 discrete R symmetries (cf. e.g. [369]).

Classification

Given the constraints on the order M, it is straightforward to classify all phenomenologically attractive charge assignments. Here we assume that the charge assignments are family blind. Though not absolutely necessary it does ensure that the symmetry does not prevent mixing between families in the fermion mass matrix. The classification was done by a scan over all possible values of M. In addition to forbidding the μ term we require that

1. Mixed gauge-$\mathbb{Z}_M^R$ anomalies cancel, i.e. $A^R_{1\le i\le 3} = \rho \mod \eta$;
2. Yukawa couplings $\mathbf{10}\,\mathbf{10}\,H_u$ and $\mathbf{10}\,\overline{\mathbf{5}}\,H_d$ as well as the neutrino mass Weinberg operator $\overline{\mathbf{5}}\,H_u\,\overline{\mathbf{5}}\,H_u$ are allowed;
3. R-parity violating couplings are forbidden.

Under these constraints the allowed charge assignments are given in Table 19.1.

For completeness we note that there are only two more charge assignments that are allowed demanding just the first two conditions. They are given in Table 19.2.

Table 19.1 Phenomenologically attractive charge assignments

M	$q_{\mathbf{10}}$	$q_{\overline{\mathbf{5}}}$	q_{H_u}	q_{H_d}	$q^{\mathrm{sh}}_{H_u}$	$q^{\mathrm{sh}}_{H_d}$	ρ	A_0^R(MSSM)
4	1	1	0	0	16	16	1	1
6	5	3	4	0	28	24	0	1
8	1	5	0	4	24	28	1	3
12	5	9	4	0	28	24	3	1
24	5	9	16	12	88	84	9	7

The charges $q^{\mathrm{sh}}_{H_u}$ and $q^{\mathrm{sh}}_{H_d}$ are Higgs charges shifted in such a way that the anomaly coefficients A_i^R $(1 \le i \le 3)$ are manifestly universal. ρ is the universal value of the anomaly coefficients; $\rho \ne 0$ indicates GS cancellation of anomalies. *Reprinted from Nuclear Physics B 850, H.M. Lee, S. Raby, M. Ratz, G.G. Ross, R. Schieren, K. Schmidt-Hoberg, and P.K.S. Vaudrevange, "Discrete R symmetries for the MSSM and its singlet extensions," Page 5, Copyright (2011), with permission from Elsevier*

Table 19.2 Charge assignments which satisfy only the first two criteria

M	$q_{\mathbf{10}}$	$q_{\overline{\mathbf{5}}}$	q_{H_u}	q_{H_d}	$q^{\mathrm{sh}}_{H_u}$	$q^{\mathrm{sh}}_{H_d}$
3	2	0	1	0	10	12
6	2	0	4	0	28	24

Both assignments have $\rho = 0$. *Reprinted from Nuclear Physics B 850, H.M. Lee, S. Raby, M. Ratz, G.G. Ross, R. Schieren, K. Schmidt-Hoberg, and P.K.S. Vaudrevange, "Discrete R symmetries for the MSSM and its singlet extensions," Page 5, Copyright (2011), with permission from Elsevier*

One may ask whether there are additional discrete symmetries, such as $\mathbb{Z}_O^{(R)} \times \mathbb{Z}_P^{(R)}$, which cannot be written as single $\mathbb{Z}_M^{(R)}$ symmetries but also fulfill the three criteria above. The only candidates for such symmetries are based on the two patterns shown in Table 19.2. We find that by amending these assignments by the usual matter parity one arrives at the $\mathbb{Z}_6^R$ symmetry of Table 19.1. Hence our classification also comprises the $\mathbb{Z}_O^{(R)} \times \mathbb{Z}_P^{(R)}$ case. Of course, in extensions of the MSSM, extra states can enjoy additional symmetries.

Dimension Five Nucleon Decay Operators

Note that the third condition is sufficient to eliminate baryon and lepton number violation due to dimension four terms in the Lagrangian density. However in the MSSM at dimension five there are problematic operators allowed that generate nucleon decay. To be consistent with the bounds on nucleon decay these must be suppressed by a mass scale more than eight orders above the Planck scale, a major problem. However in the case of the $\mathbb{Z}_M^R$ symmetries these operators are automatically absent. To see this note that the requirement that up- and down-type Yukawa couplings be allowed implies

$$3q_{\mathbf{10}} + q_{\overline{\mathbf{5}}} + q_{H_u} + q_{H_d} \;=\; 4 \mod M\ . \tag{19.12}$$

Combining this with Eq. (19.8a) gives

$$3q_{\mathbf{10}} + q_{\overline{\mathbf{5}}} \;=\; 0 \mod M\ , \tag{19.13}$$

showing that (for $M \neq 2$) the troublesome dimension five operators $\mathbf{10}\,\mathbf{10}\,\mathbf{10}\,\overline{\mathbf{5}}$ are automatically forbidden whenever the Yukawa couplings are allowed.

The Gravitational Anomaly Constraint

For all charge assignments, the MSSM contribution to the gravitational anomaly is

$$A_0^R(\mathrm{MSSM}) \;=\; 7 \mod \eta\ . \tag{19.14}$$

All cases except for $M = 6$ have $\rho \neq 0$ and hence require the presence of an axion a. Call the multiplet containing the axion S,

$$S|_{\theta=0} \;=\; s + \mathrm{i}\,a\ ; \tag{19.15}$$

later we will identify S with the dilaton. From the coupling to the gauge fields $\int \mathrm{d}^2\theta\, S\, W_\alpha\, W^\alpha$ one infers that the axino/dilatino has R charge -1. Therefore, after adding the axino/dilatino contribution we obtain

$$A_0^R(\text{MSSM} + \text{axino/dilatino}) \;=\; 6 \mod \eta\,. \tag{19.16}$$

The condition for anomaly freedom is

$$\frac{1}{24}\left(A_0^R \mod \eta\right) \;=\; A_i^R \mod \eta \tag{19.17}$$

for $1 \leq i \leq 3$. Now, since $A_i^R \in \mathbb{Z}$ and since the order M, and therefore η, divides 24, this condition is equivalent to

$$A_0^R \;=\; 0 \mod \eta\,. \tag{19.18}$$

From Eq. (19.16) we see that the cases $M = 4$ and 12 are anomaly free. The case $M = 6$ is anomaly free with an axion that is singlet under $\mathbb{Z}_6^R$. All the other cases require additional states in order to cancel the gravitational anomaly.

However this does not necessarily require additional states in the low energy spectrum. This is because states contributing to the anomaly can acquire mass when the symmetry is spontaneously broken. Since the R symmetry is broken in the hidden sector when supersymmetry is broken these states can acquire a mass of order the supersymmetry breaking scale in the hidden sector which can be as large as 10^{13} GeV. With this in mind we will not consider the gravitational anomaly any further.

19.2 A Simple $\mathbb{Z}_4^R$ Symmetry in the MSSM

In Table 19.1 we survey all symmetries and charge assignments which commute with SU(5). The simplest one, the $\mathbb{Z}_4^R$, commutes also with SO(10). In what follows we will discuss this case in more detail. This symmetry was first considered in [353, 362].

Non-perturbative Terms

The gauge invariant superpotential of the MSSM contains

$$\begin{aligned}\mathscr{W} &= \mu\, H_u\, H_d + \kappa_i\, L_i H_u \\ &\quad + Y_e^{ij} H_d\, L_i\, \overline{E}_j + Y_d^{ij} H_d\, Q_i\, \overline{D}_j + Y_u^{ij} H_u\, Q_i\, \overline{U}_j \\ &\quad + \lambda_{ijk}^{(0)}\, L_i L_j \overline{E}_k + \lambda_{ijk}^{(1)}\, L_i Q_j \overline{D}_k + \lambda_{ijk}^{(2)}\, \overline{U}_i\, \overline{D}_j\, \overline{D}_k\end{aligned}$$

$$
\begin{aligned}
&+\kappa_{ij}^{(0)}\, H_u\, L_i\, H_u\, L_j + \kappa_{ijk\ell}^{(1)}\, Q_i\, Q_j\, Q_k\, L_\ell + \kappa_{ijk\ell}^{(2)}\, \overline{U}_i\, \overline{U}_j\, \overline{D}_k\, \overline{E}_\ell \\
&+\kappa_{ijk}^{(3)}\, Q_i\, Q_j\, Q_k\, H_d + \kappa_{ijk}^{(4)}\, Q_i\, \overline{U}_j\, \overline{E}_k\, H_d + \kappa_i^{(5)} L_i H_u H_u H_d \,. \qquad (19.19)
\end{aligned}
$$

We see immediately that the coefficients μ, κ_i, $\lambda_{ijk}^{(0)}$, $\lambda_{ijk}^{(1)}$, $\lambda_{ijk}^{(2)}$, $\kappa_{ijk\ell}^{(1)}$, $\kappa_{ijk\ell}^{(2)}$, $\kappa_{ijk}^{(3)}$, $\kappa_{ijk}^{(4)}$ and $\kappa_i^{(5)}$ are forbidden by $\mathbb{Z}_4^R$ perturbatively while $Y_{e,d,u}^{ij}$ and $\kappa_{ij}^{(0)}$ are allowed. In what follows we will show that at the non-perturbative level μ as well as $\kappa_{ijk\ell}^{(1)}$ and $\kappa_{ijk\ell}^{(2)}$ will be induced while the R parity violating couplings κ_i and λ as well as the $\kappa^{(3-5)}$ remain zero. The reason is that the latter are forbidden by a $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_4^R$ which is equivalent to matter parity. This subgroup is unbroken by the supersymmetry breaking sector and thus remains a symmetry of the full theory.

Let us spell out the argument in somewhat more detail. Call the $\mathbb{Z}_4^R$ transformation ζ,

$$
\begin{aligned}
\zeta \;:\; \text{matter superfield} &\to \mathrm{i}\cdot \text{matter superfield}\,, \\
\text{Higgs superfield} &\to \text{Higgs superfield}\,, \\
\theta &\to \mathrm{i}\cdot\theta\,, \\
\mathscr{W} &\to -\mathscr{W}\,.
\end{aligned} \qquad (19.20)
$$

Now look at the transformation ζ^2, under which matter superfields transform with a minus, Higgs superfields go into themselves and $\theta \to -\theta$. The transformation fermion $\to$ $-$fermion and $\theta \to -\theta$ is a symmetry of any SUSY theory, therefore ζ^2 is equivalent to matter parity, and, in particular, anomaly free with $\rho = 0$. One can use the path integral (cf. Section "Appendix: Discrete Anomalies in the Path Integral Approach") to show that correlators that vanish due to a non-anomalous symmetry with $\rho = 0$ also vanish at the quantum level. Therefore, the matter parity subgroup contained in the $\mathbb{Z}_4^R$ will not be violated by quantum effects.

On the other hand, correlators which are only forbidden by $\mathbb{Z}_4^R$ but not by $\mathbb{Z}_2$, i.e. which are invariant under ζ^2, can be non-trivial at the quantum level. A convenient way to parametrize effective couplings describing these effects involve the S field, which shifts under the $\mathbb{Z}_4^R$ symmetry as [cf. Eq. (19.61)]

$$
S \;\to\; S + \frac{\mathrm{i}}{2}\Delta_{\mathrm{GS}}\,. \qquad (19.21)
$$

The discrete shift of S is given by [cf. Eq. (19.62)]

$$
\Delta_{\mathrm{GS}} \;=\; \frac{1}{4\pi}(A_{G-G-\mathbb{Z}_4^R} \mod 2) \;=\; \frac{1+2\nu}{4\pi} \qquad (19.22)
$$

with $\nu \in \mathbb{Z}$. This allows us to write down terms

$$\Delta\mathscr{W}_{\mathrm{np}} = \exp\left(-8\pi^2 \frac{1+2n}{1+2\nu} S\right) \Big[B_0 + \overline{\mu}\, H_u\, H_d + \overline{\kappa}^{(1)}_{ijk\ell}\, Q_i\, Q_j\, Q_k\, L_\ell + \overline{\kappa}^{(2)}_{ijk\ell}\, \overline{U}_i\, \overline{U}_j\, \overline{D}_k\, \overline{E}_\ell\Big] \tag{19.23}$$

with some coefficients B_0, $\overline{\mu}$ and $\overline{\kappa}^{(1,2)}_{ijk\ell}$ and $n \in \mathbb{Z}$. Such superpotential terms are $\mathbb{Z}_4^R$ covariant, i.e. the exponential transforms with a minus under $\mathbb{Z}_4^R$ while the terms in the square brackets are invariant. Due to the fact that S enters the gauge kinetic function, these terms are proportional to $\mathrm{e}^{-8\pi^2 \frac{1+2n}{1+2\nu} \frac{1}{g^2}}$. For $n = \nu = 0$ they can be interpreted as originating from t'Hooft instanton effects. The $8\pi^2$ in the exponential can also be obtained directly in a stringy computation [276]. The crucial property of the non-perturbative couplings (19.23) is that they are naturally suppressed.

The critical question concerns now the interpretation of the $\mathrm{e}^{-8\pi^2 \frac{1+2n}{1+2\nu} S}$ terms. So far we have shown that such terms are $\mathbb{Z}_4^R$ covariant. In the MSSM as a 'stand-alone' theory, $\mathrm{SU}(3)_C$ or $\mathrm{SU}(2)_\mathrm{L}$ instantons can generate such terms, but their magnitude turns out to be very small. Whether or not further terms, with given n and ν, appear depends on the model. Let us now make the very common assumption that there is a hidden sector that gets strong at some intermediate scale Λ. Then the non-perturbative terms related to the strong dynamics may well be the source of supersymmetry breakdown [12, 370]. Given non-renormalizable interactions between the MSSM and the hidden sector, communicated by some messenger fields, Λ sets the magnitude of the MSSM soft terms, $m_{\mathrm{soft}} \sim \Lambda^3/M_*^2$, with M_* being the messenger scale. In such settings, holomorphic, i.e. superpotential, terms can also be induced by higher-dimensional operators. That is, the $\Delta\mathscr{W}_{\mathrm{np}}$ terms can appear with magnitude $m_{\mathrm{soft}} \sim \Lambda^3/M_*^2$, but in principle they may also be absent if there are no higher-dimensional operators connecting the MSSM sector with the hidden sector exhibiting strong dynamics. In other words, if the MSSM fields are singlets under the hidden sector gauge interactions, there is, a priori, no guarantee that the $\Delta\mathscr{W}_{\mathrm{np}}$ terms appear with reasonable size. If the scale of MSSM soft terms is related to some hidden sector strong dynamics, we expect the holomorphic terms also to appear, unless there are additional symmetries beyond $\mathbb{Z}_4^R$ that forbid such couplings. Assuming that the dominant non-perturbative scale is related to supersymmetry breakdown we expect that the $\Delta\mathscr{W}_{\mathrm{np}}$ terms are of the order of the soft supersymmetry breaking terms. We will mainly focus on gravity mediation, where $M_* = m_{Pl}$ and these terms are of the order of the gravitino mass $m_{3/2}$ (in Planck units). Below in Chap. 23 we will present an explicit string theory example in which the non-perturbative μ term is directly connected to $m_{3/2}$.

At this point let us mention that for the case of discrete R symmetries we disagree with statements made in [369], where it was claimed that, in the context of gravity mediation, R symmetries will be broken at the Planck scale and be therefore ineffective. The claim relies on the observation that there are fields with Planck scale VEVs that break the R symmetry. The derivation of this result relies on the inequality $|\langle \mathscr{W} \rangle| \le \frac{1}{2} f_r |F|$ (cf. Eq. (9) in [369]) where f_r is the R-axion decay constant. This was derived for the case of continuous R symmetries by taking the limit of an infinitesimal transformation [369]. For the case of discrete R symmetries the inequality is no longer true and there is no requirement that R-non singlets acquire Planck scale VEVs. In this case the R symmetry can be broken at a much lower scale. This is the case in the supergravity examples discussed here. In them the breaking of the R symmetry occurs non-perturbatively at an intermediate scale in a hidden sector and it is the superpotential VEV $\langle \mathscr{W} \rangle$ rather than a field VEV that is the order parameter for R symmetry breaking. Since the superpotential only appears at the non-perturbative level it is small. Also all other R symmetry breaking terms are small. This applies also to other schemes such as the one discussed in [371], where a small $\langle \mathscr{W} \rangle$ is a consequence of an approximate R symmetry. Here the R symmetry is broken perturbatively, but again the order parameter, i.e. the superpotential VEV, is very small. In conclusion, R symmetries are a useful tool also, or in particular, in gravity mediation, where the same parameter, the small superpotential VEV, both sets the scale of soft masses and cancels the vacuum energy. In what follows, we discuss how the connection between the $\Delta\mathscr{W}_{\mathrm{np}}$ terms and $m_{3/2}$ arises in the scheme of Kähler stabilization.

Dilaton Stabilization and Supersymmetry Breaking

At the present stage of the discussion, the S field has no potential and supersymmetry is unbroken. An economical way to rectify this situation is to invoke the stringy scheme of Kähler stabilization [372–375].[3] In this case the term of the form e^{-bS} represents a hidden sector gaugino condensate [370], which sets the scale for supersymmetry breakdown. According to the above discussion, in the presence of our $\mathbb{Z}_4^R$ symmetry

$$b \;=\; 8\pi^2\,\frac{1+2n}{1+2\nu}\;. \tag{19.24}$$

Let us discuss what that means in the case of a hidden $\mathrm{SU}(N_c)$ theory with N_f chiral superfields in the $\boldsymbol{N_c} + \overline{\boldsymbol{N_c}}$ representations. Here the coefficient b is given by

$$b \;=\; \frac{3}{2\beta} \;=\; \frac{3\cdot 8\pi^2}{3N_c - N_f}\;. \tag{19.25}$$

[3] Alternatively, other stabilization schemes, such as racetrack mechanisms, may be applicable here.

Therefore

$$\frac{3}{3N_c - N_f} = \frac{1+2n}{1+2\nu} . \tag{19.26}$$

In the scheme under consideration, supersymmetry is broken by a non-trivial VEV of F_S. This leads to gaugino and soft scalar masses, following the pattern of the so-called "dilaton dominated scenario" [376]. This scenario has a number of phenomenologically attractive features. In particular, due to flavour universality in the soft breaking sector, it avoids the SUSY FCNC problem. Also, most of the physical CP phases, e.g. $\arg(A^* M)$, vanish which ameliorates the SUSY CP problem. However in the dilaton dominated case the vacuum structure may favour an unacceptable colour breaking minimum [377]. Other phenomenological aspects have been discussed in [378].

Moreover, the (non-perturbative) superpotential acquires a non-trivial VEV as well,

$$\langle \mathscr{W} \rangle \sim \mathrm{e}^{-b\,\langle S \rangle} \neq 0 . \tag{19.27}$$

All gauge invariant terms which have been forbidden because they have zero R charge can now be obtained by multiplying them with $\langle \mathscr{W} \rangle$. $\langle \mathscr{W} \rangle$ will hence be the order parameter for R symmetry breaking. Inserting this in Eq. (19.23) we find that there will be a μ term of the order of $\langle \mathscr{W} \rangle$, i.e. of the order of the gravitino mass $m_{3/2}$, as well as $\kappa^{(1)}_{ijk\ell} \sim 10^{-15}/M_\mathrm{P}$. On the other hand, terms which have odd $\mathbb{Z}_4^R$ charge cannot be obtained by multiplying them by $\mathrm{e}^{-b\,\langle S \rangle}$; these are precisely the R parity violating couplings κ_i, $\lambda^{(0)}$, $\lambda^{(1)}$ and $\lambda^{(2)}$ in Eq. (19.19), showing again that matter parity will not be broken.

Phenomenology

The suppression of the $\kappa^{(1)}$ term leads to a situation in which dimension five proton decay will be unobservably small. Therefore, proton decay will proceed through dimension six operators mediated by gauge boson exchange.

In settings with discrete R symmetries one should worry about the cosmological domain wall problem [379]. The domain walls form at the stage of R symmetry breaking, typically the scale of supersymmetry breaking. For the case of gravity mediation this is at an intermediate scale of $\mathscr{O}(10^{12})$ GeV. Provided the Hubble scale during inflation is below this scale, domain walls have sufficient time to form and then they will be inflated away. The requirement that no domain walls are created after inflation translates in an upper bound on the reheat temperature T_R, which, given the other bounds on T_R in supersymmetric cosmology, appears rather mild.

A discrete R symmetry may also be useful for inflationary scenarios. For example, in [380], it is argued that a $\mathbb{Z}_8^R$ symmetry, with inflaton field ϕ carrying R charge 2, can be used to guarantee that the inflaton potential is flat near the origin and give enough inflation.[4]

In summary, for the case of gravity mediated supersymmetry breaking, non-perturbative effects naturally generate a μ parameter of the order of the gravitino mass. The symmetry ensures that the proton decay rate is well below the experimental limit and an exact matter parity is left that guarantees SUSY particles can only be pair produced and the lightest SUSY particle is stable. Thus one is left with the usual MSSM phenomenology with negligibly small corrections from higher dimension terms.

19.3 Singlet Extensions

In Sect. 19.1.1 we have shown that the requirement of universality for the mixed gauge anomalies constrains the order M of a potential $\mathbb{Z}_M^R$ symmetry to be a divisor of 24. As we have seen, this analysis carries over in an obvious way to singlet extensions of the MSSM, since additional SM singlet fields cannot change the constraints coming from the mixed gauge anomalies. In such extensions the MSSM subsector still has to obey the criteria derived in Sect. 19.1.1. However, the extra (singlet) fields can be subject to additional symmetries.

In what follows we concentrate on one simple singlet extension, i.e. the so-called NMSSM, in which the singlet couples to the Higgs bilinear and there are cubic self-interactions. A second singlet extension which solved the strong CP problem can be found in [364].

NMSSM

In the NMSSM, there is one additional singlet N with superpotential

$$\mathscr{W} \; = \; \mathscr{W}_{\mathrm{MSSM}}^{\mu=0} + \lambda\, N H_u H_d + \kappa\, N^3 \; . \tag{19.28}$$

Let us now consider what this implies for the order M of a $\mathbb{Z}_M^R$ symmetry.

[4]Note, their definition of the order of the discrete symmetry differs from ours. What they call $\mathbb{Z}_4^R$ we call $\mathbb{Z}_8^R$.

Constraints from NMSSM Couplings

There are three different classes of $\mathbb{Z}_M^R$ symmetries for which the N^3-term of Eq. (19.28) implies different charges for the singlet N, i.e.

$$M = 0 \bmod 3 \quad \Rightarrow \quad \text{no } N^3 \text{ term possible} , \tag{19.29}$$

$$M = 1 \bmod 3 \quad \Rightarrow \quad q_N = \frac{M+2}{3} \bmod M , \tag{19.30}$$

$$M = 2 \bmod 3 \quad \Rightarrow \quad q_N = \frac{2M+2}{3} \bmod M , \tag{19.31}$$

with q_N the $\mathbb{Z}_M^R$ charge of N.

$M = 1$ mod 3

Let us first consider the case $M = 1 \bmod 3$. The term $\lambda N H_u H_d$ together with Eq. (19.8a) then implies

$$\left(\frac{M+2}{3} \bmod M\right) + (4 \bmod 2\eta) = 2 \bmod M$$
$$\Rightarrow \frac{M+8}{3} = 0 \bmod M . \tag{19.32}$$

This equation has only one non-trivial solution for integer M, namely $M = 4$. Note that in this case $q_N = 2 \bmod 4$ and a linear term in N is also allowed in the superpotential. Strictly speaking this is not the NMSSM but it is viable if the linear term is very small. We will discuss later why this may be natural.

Following the analysis of Sect. 19.1.1 and using Eq. (19.30), the unique charge assignment compatible with the Weinberg operator is shown in Table 19.3.

Table 19.3 Charge assignments for the $\mathbb{Z}_4^R$ symmetry

M	$q_{\mathbf{10}}$	$q_{\overline{\mathbf{5}}}$	q_{H_u}	q_{H_d}	q_N
4	1	1	0	0	2

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This is exactly the $\mathbb{Z}_4^R$ symmetry which we discussed in Sect. 19.2. We have seen that the mixed gauge anomaly coefficients of this symmetry satisfy the Green–Schwarz condition. Of course the singlet does not change these coefficients, so the analysis still applies.

$M = 2$ mod 3

Let us now consider the case $M = 2 \mod 3$. The term $\lambda N H_u H_d$ together with Eq. (19.8a) then implies

$$\frac{2M+8}{3} = 0 \bmod M\ . \tag{19.33}$$

The solutions to this equation are $M = 2, 8$. As we have noted earlier there are no meaningful $M = 2$ R symmetries. The $M = 8$ case however is very interesting since, in this case, $q_N = 6 \mod 8$ and the linear term in N is forbidden. Following the analysis of Sect. 19.1.1 and using Eq. (19.31), the unique charge assignment compatible with the Weinberg operator is shown in Table 19.4.

As the singlet does not contribute to mixed gauge anomalies, we know already from Table 19.1 that the $\mathbb{Z}_8^R$ symmetry has $A^R_{1\le i\le 3}(\mathrm{MSSM}) = \rho = 1$.

The Hierarchy Problem

Searching for possible $\mathbb{Z}_M^R$ symmetries in the context of the NMSSM we found that there are only two potential candidates: a $\mathbb{Z}_4^R$ and a $\mathbb{Z}_8^R$ symmetry. The $\mathbb{Z}_4^R$ symmetry is actually a subgroup of the $\mathbb{Z}_8^R$ symmetry, hence both symmetries are closely related. While the $\mathbb{Z}_4^R$ commutes with SO(10) the $\mathbb{Z}_8^R$ only commutes with SU(5). In both cases all dimension four and five baryon and lepton number violating operators are forbidden (except for the Weinberg operator), consistent with what we found in Sect. 19.1.1.

Table 19.4 Charge assignments for the $\mathbb{Z}_8^R$ symmetry

M	$q_{\mathbf{10}}$	$q_{\overline{\mathbf{5}}}$	q_{H_u}	q_{H_d}	q_N
8	1	5	0	4	6

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A potential problem with NMSSM models arises because SUSY breaking breaks the R symmetry and in radiative order a linear term in N is generated in the superpotential. If the coefficient of this linear term is larger than the square of the electroweak scale it will lead to a large VEV for the singlet N and therefore to a destabilization of the SUSY solution to the gauge hierarchy problem. This has been studied in detail by Abel [381] who showed that the only dangerous operators that induce divergent tadpoles arise either from even terms in the superpotential or odd terms in the Kähler potential. He also showed that an R symmetry can avoid such terms because of the different R charges of the super- and Kähler potential (cf. also [382]). From the charge assignments of Tables 19.3 and 19.4 for the singlet N and the Higgs fields it is easy to show that the super- and Kähler-potentials actually do have exactly this structure in both the $\mathbb{Z}_4^R$ and the $\mathbb{Z}_8^R$ case and so in both cases radiative corrections do not destabilise the SUSY solution to the hierarchy problem.

The main difference between $\mathbb{Z}_4^R$ and $\mathbb{Z}_8^R$ is that the former allows a linear term even at tree level. Does this mean that it is necessary to have the full $\mathbb{Z}_8^R$ symmetry when building the NMSSM? In effective theories, such as those describing the massless degrees of freedom in string compactifications, the superpotential starts with cubic terms in the fields and the linear term only appears through the coupling of the singlet field to fields acquiring VEVs. If the only (non-moduli) fields, ϕ, with VEVs above the electroweak scale are in the hidden sector the coupling will be suppressed by messenger field masses, M_*, which may be as large as the Planck scale. Allowing for trilinear couplings to messenger fields as well as trilinear couplings between messenger and hidden sector fields and assuming no additional symmetries, the leading term in the superpotential after integrating out the messenger fields is $N\phi^4/M_*^2$ with the messenger scale M_*. Taking Planck scale messengers, the constraint that this should not disturb the hierarchy is that $\langle\phi\rangle \leq \sqrt{M_W M_\mathrm{P}}$ which is satisfied if the dominant VEV comes from the SUSY breaking sector. In this case it is sufficient to impose just the $\mathbb{Z}_4^R$ symmetry when building the NMSSM.

The role of the SM singlets $\psi_2^{(i)}$ with R-charge 2 (such as N for the case $\mathbb{Z}_4^R$) has recently been discussed in the context of singlet (moduli) stabilization [276]. There it was found that for a superpotential with generic coefficients the number of singlets with R-charge 2 should not exceed the number of fields $\phi_0^{(j)}$ with R-charge 0 since otherwise the F-term conditions would overconstrain the system. Moreover, the $\psi_2^{(i)}$ fields pair up with an equal number of $\phi_0^{(j)}$ fields. That is, for generic superpotential coefficients one might not expect to find vacua with an unbroken $\mathbb{Z}_4^R$ symmetry and a massless singlet with R-charge 2. However, it is quite conceivable that there are symmetries between the F-terms. In such a situation the $\psi_2^{(i)} - \phi_0^{(j)}$ mass matrix won't have full rank such that one is effectively left with one (or more) singlet(s) with R-charge 2. It will be interesting to see if this situation can be realized in string models in which there are additional symmetries, such as D_4 [383, 384], relating the superpotential coefficients.

Non-perturbative Effects

Non-perturbative effects may also be important in determining the low energy phenomenology. From Eq. (19.23) we see that the superpotential has a term of the form

$$\Delta\mathscr{W}_{\mathrm{np}} = B_0\,\mathrm{e}^{-bS} \tag{19.34}$$

with a constant b. This parametrizes the non-perturbative effects discussed above, and may be interpreted as a hidden sector gaugino condensate. It provides the order parameter for local supersymmetry and generates the gravitino mass

$$\frac{\langle\Delta\mathscr{W}_{\mathrm{np}}\rangle}{M_{\mathrm{P}}^2} \sim \frac{\langle\lambda\lambda\rangle}{M_{\mathrm{P}}^2} \sim m_{3/2}\ . \tag{19.35}$$

$\Delta\mathscr{W}_{\mathrm{np}}$ has R-charge 2 (cf. the discussion in Sect. 19.2) and similar non-perturbative effects can contribute to further terms in the superpotential. The crucial property of the non-perturbative couplings is that they are naturally suppressed. To parametrize these effects we denote by a superfield Y a non-perturbative term of the form given in Eq. (19.34) (scaled by the factor M_{P}^{-2}) carrying R-charge 2 and we construct the superpotential involving Y that is consistent with the relevant R symmetry.

The lowest superpotential terms in Y have the form

$$\begin{aligned}\Delta\mathscr{W}_{\mathbb{Z}_4^R} &= Y + Y^2 N + Y N^2 + Y H_u H_d \\ &\sim m_{3/2} M_{\mathrm{P}}^2 + m_{3/2}^2 N + m_{3/2} N^2 + m_{3/2} H_u H_d\ ,\end{aligned} \tag{19.36}$$

$$\begin{aligned}\Delta\mathscr{W}_{\mathbb{Z}_8^R} &= Y + Y^2\left(N + Y N^2 + Y H_u H_d\right) \\ &\sim m_{3/2} M_{\mathrm{P}}^2 + m_{3/2}^2 N + \frac{m_{3/2}^3}{M_{\mathrm{P}}^2} N^2 + \frac{m_{3/2}^3}{M_{\mathrm{P}}^2} H_u H_d\ .\end{aligned} \tag{19.37}$$

All of these terms have magnitude determined by the gravitino mass scale. For gauge mediation this scale can be very small and these terms negligible. For gravity mediation however the gravitino mass scale is the scale of supersymmetry breaking in the visible sector and the unsuppressed terms cannot be neglected. In this case, the magnitude of the $\Delta\mathscr{W}_{\mathbb{Z}_4^R}$ terms is such as to reproduce the superpotential of the S-MSSM [385, 386], where, apart from the usual NMSSM couplings also holomorphic mass terms for the singlets and the Higgs fields of the order $m_{3/2}$ are introduced. This extension of the SM has been shown to significantly reduce the fine tuning needed to accommodate the LEP Higgs mass bound [385, 386]. This general NMSSM [GNMSSM] has been further studied in [288, 387, 388] where they have shown that it minimizes the amount of fine-tuning. Our analysis yields a justification for the small holomorphic terms, which have so far just been imposed by hand.

Interestingly the form of the non-perturbative effects is very sensitive to the underlying symmetry. For the case of $\mathbb{Z}_4^R$ there are additional unsuppressed linear and quadratic terms in N as well as a non-perturbative contribution to the Higgsino mass. For the case of $\mathbb{Z}_8^R$ only the linear term in N is unsuppressed. Because the magnitude of all these terms is determined by the gravitino mass they will not disturb the SUSY solution to the hierarchy problem. However, for the case of gravity mediation, the terms cannot be neglected and may be expected to significantly change the NMSSM phenomenology. Given the different non-perturbative terms appearing in the $\mathbb{Z}_4^R$ and $\mathbb{Z}_8^R$ we may expect these to have different phenomenological implications.

19.4 Conclusions

We have discussed possible discrete symmetries for the MSSM which commute with SU(5). We have seen that, in order to address the μ problem, these have to be R symmetries. We have surveyed all possible discrete $\mathbb{Z}_M^R$ symmetries. Anomaly cancellation requires that the order M be a divisor of 24. We identified 5 phenomenologically viable symmetries for the MSSM.

The simplest of the 5 MSSM symmetries is a $\mathbb{Z}_4^R$ which commutes with SO(10). This symmetry forbids all R-parity violating couplings, dimension five proton decay operators and the μ term at tree-level while allowing the usual Yukawa couplings and the neutrino mass operator. At the non-perturbative level the μ term and the dimension five proton decay operators are generated. We argued that in settings in which supersymmetry breaking is related to some non-perturbative dynamics the μ term will be of the order of the MSSM soft terms. In particular, in gravity mediation we will have $\mu \sim m_{3/2}$ and coefficients of the dimension five proton decay operators $\kappa_{ijk\ell}^{(1,2)} \sim m_{3/2}/M_\mathrm{P}^2$, i.e. sufficiently suppressed. Thus the $\mathbb{Z}_4^R$ symmetry provides us with a simultaneous solution to the arguably two most severe problems of the MSSM.

We have discussed the role of discrete symmetries in singlet extensions of the MSSM. There are two possible symmetries consistent with the structure of the NMSSM, $\mathbb{Z}_4^R$ and $\mathbb{Z}_8^R$, both of which are capable of solving the hierarchy problem. The $\mathbb{Z}_8^R$ allows the usual couplings while forbidding the linear term for the singlet at the perturbative level. In the $\mathbb{Z}_4^R$ case, one obtains holomorphic mass terms for the singlet and the Higgs at the non-perturbative level. We have argued that the size of such terms is of the order $m_{3/2}$, leading to an GNMSSM-like scheme in which the smallness of the explicit mass terms for the singlets and Higgs finds an explanation.

Finally, in Chap. 23 we describe how to embed the $\mathbb{Z}_4^R$ into string theory. Specifically, in [276], a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with this $\mathbb{Z}_4^R$ was constructed with the exact MSSM spectrum below the compactification scale. The $\mathbb{Z}_4^R$ originates from the Lorentz symmetry of compactified dimensions. At the non-perturbative level the μ term may be generated as a consequence of the $\mathbb{Z}_4^R$ anomaly. There is an exact matter parity and dimension five proton decay is well below experimental limits.

Appendix: Discrete Anomalies in the Path Integral Approach

In this appendix we re-derive Abelian discrete anomalies with the path integral method, following [389, 390]. Among other things, we will describe how this allows us to understand the discrete version of the Green–Schwarz mechanism.

Path Integral Derivation of Anomalies

Consider a theory described by a Lagrangian density $\mathscr{L}$ with a set of fermions $\Psi = [\psi^{(1)}, \dots, \psi^{(M)}]$, where $\psi^{(m)}$ denotes a field transforming in the irreducible representation (irrep) $\boldsymbol{R}^{(m)}$ of all internal symmetries. A general transformation $\Psi \to U\Psi$ or, more explicitly,

$$\begin{bmatrix} \psi^{(1)} \\ \vdots \\ \psi^{(M)} \end{bmatrix} \to \begin{pmatrix} U^{(1)} & & 0 \\ & \ddots & \\ 0 & & U^{(M)} \end{pmatrix} \begin{bmatrix} \psi^{(1)} \\ \vdots \\ \psi^{(M)} \end{bmatrix} , \tag{19.38}$$

which leaves $\mathscr{L}$ invariant (up to a total derivative) denotes a classical symmetry.

A classical symmetry implies that certain correlators vanish at the classical level. To see this, consider the correlator

$$C_{n_1\dots n_M} = \left\langle (\psi^{(1)})^{n_1} \cdots (\psi^{(M)})^{n_M} \right\rangle . \tag{19.39}$$

Now, if the field combination $(\psi^{(1)})^{n_1} \cdots (\psi^{(M)})^{n_M}$ is not invariant under the symmetry transformation, we arrive at the (premature) conclusion that $C_{n_1\dots n_M} = 0$.

Classical chiral symmetries can be broken by quantum effects, i.e. have an anomaly. Specifically, consider a chiral transformation

$$\Psi(x) \to \Psi'(x) = \exp\big(2\,\mathrm{i}\,\alpha\, P_\mathrm{L}\big)\, \Psi(x) , \tag{19.40}$$

where $\alpha = \alpha^{\mathrm{anom}} \mathsf{T}_{\mathrm{anom}}$ with $\mathsf{T}_{\mathrm{anom}}$ denoting the generator of the transformation and α^{anom} being a parameter, and P_L is the left-chiral projector.

We wish now to show that this implies that vanishing correlators at the classical level may appear at the quantum level. To this end, write the correlator as a path integral,

$$C_{n_1\dots n_M} \;=\; \int \mathscr{D}\Psi\,\mathscr{D}\overline{\Psi}\,(\psi^{(1)})^{n_1}\cdots(\psi^{(M)})^{n_M}\,\mathrm{e}^{\mathrm{i}\,S}\;, \tag{19.41}$$

where S denotes the action, which is left invariant under (19.40). Now recall that under the transformation (19.38) the path integral measure undergoes a non-trivial change [391, 392],

$$\mathscr{D}\Psi\,\mathscr{D}\overline{\Psi}\;\to\;J(\alpha)\,\mathscr{D}\Psi\,\mathscr{D}\overline{\Psi}\;, \tag{19.42}$$

where the Jacobian of the transformation is given by

$$J(\alpha)\;=\;\exp\left\{\mathrm{i}\int \mathrm{d}^4x\,\mathscr{A}(\alpha)\right\}\;. \tag{19.43}$$

The crucial observation is that in the presence of a non-trivial Jacobian the full quantum correlator can be invariant. This is true regardless of whether the transformation (19.38) is continuous or discrete, or whether it is gauged or global.

The anomaly function $\mathscr{A}$ appearing in (19.43) decomposes into a gauge and a gravitational part [393–395],

$$\mathscr{A}(\alpha)\;=\;\mathscr{A}_{\mathrm{gauge}}(\alpha)+\mathscr{A}_{\mathrm{grav}}(\alpha)\;, \tag{19.44}$$

with

$$\mathscr{A}_{\mathrm{gauge}}(\alpha)=\frac{1}{16\,\pi^2}\,\mathrm{Tr}\,\left[\alpha\,\mathscr{F}\widetilde{\mathscr{F}}\right]\;, \tag{19.45}$$

$$\mathscr{A}_{\mathrm{grav}}(\alpha)=-\frac{1}{192\,\pi^2}\mathscr{R}\widetilde{\mathscr{R}}\;\mathrm{Tr}\;[\alpha]\;. \tag{19.46}$$

We have suppressed index contractions, i.e. $\mathscr{F}\widetilde{\mathscr{F}}=\mathscr{F}^{\mu\nu}\,\widetilde{\mathscr{F}}_{\mu\nu}$. Here $\mathscr{F}_{\mu\nu}=[D_\mu,D_\nu]$ is the field strength of the gauge symmetry, such that $\mathscr{F}_{\mu\nu}=(\partial_\mu A_\nu-\partial_\nu A_\mu)$ for a U(1) symmetry, $\mathscr{F}_{\mu\nu}=F^a_{\mu\nu}\,\mathsf{T}_a$ for non-Abelian gauge groups, and $\widetilde{\mathscr{F}}^{\mu\nu}=\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathscr{F}_{\rho\sigma}$ denotes its dual. Similarly, $\mathscr{R}$ represents the Riemann curvature tensor and $\mathscr{R}\widetilde{\mathscr{R}}=\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathscr{R}_{\mu\nu}{}^{\lambda\gamma}\mathscr{R}_{\rho\sigma\lambda\gamma}$. The trace 'Tr' runs over all internal indices. For a non-Abelian gauge theory the factor

$$\frac{1}{16\pi^2}\int d^4xTr(\mathscr{F}^{\mu\nu}\,\widetilde{\mathscr{F}}_{\mu\nu})=\frac{1}{32\pi^2}\int d^4x(\mathscr{F}^{\mu\nu a}\,\widetilde{\mathscr{F}}^a_{\mu\nu})\in\mathbb{Z} \tag{19.47}$$

with the convention $Tr(T_a\,T_b)=\frac{1}{2}\,\delta_{ab}$.

Now we specialize to the case where α is a $\mathbb{Z}_M$ transformation given by $\alpha = \frac{2\pi}{M}\, q^{(f)}$ with integer values of $q^{(f)}$.

$$J(\alpha) = \exp\left(i\, 2\pi \sum_{\boldsymbol{r}^{(f)}} \frac{(2\,\ell\left(\boldsymbol{r}^{(f)}\right)\; q^{(f)})}{M}\, \frac{1}{32\,\pi^2} \int d^4x (\mathscr{F}^{\mu\nu a}\, \widetilde{\mathscr{F}}^{a}_{\mu\nu}) \right) . \tag{19.48}$$

For the anomaly to be absent, i.e. $J(\alpha) = 1$, we arrive at the conditions [389, 390]

$$G - G - \mathbb{Z}_M : \quad \sum_{\boldsymbol{r}^{(f)}} \ell\left(\boldsymbol{r}^{(f)}\right)\, q^{(f)} \;=\; 0 \mod \eta\,, \tag{19.49a}$$

$$\mathrm{grav} - \mathrm{grav} - \mathbb{Z}_M : \sum_m q^{(m)} \;=\; 0 \mod \eta\,, \tag{19.49b}$$

where [cf. Eq. (19.3)]

$$\eta \;=\; \begin{cases} M & \text{for } M \text{ odd}\,, \\ M/2 & \text{for } M \text{ even} \end{cases}$$

and $q^{(m)}$ denotes the $\mathbb{Z}_M$ charge. The first sum runs over all irreducible representations $\boldsymbol{r}^{(f)}$ of G with Dynkin index[5] $\ell(\boldsymbol{r}^{(f)})$ while the second sum runs over all fermions. Our conventions are such that $\ell(\boldsymbol{N}) = 1/2$ for SU(N) and $\ell(\boldsymbol{N}) = 1$ for SO(N). Equation (19.49) are the traditional discrete anomaly conditions [89, 359] with the difference that the $\mathbb{Z}_M^3$ constraints do not appear. The issue of $\mathbb{Z}_M^3$ anomalies is discussed in [364]. It is argued that they address the discrete symmetry only to the extent that it is obtained by embedding into a $U(1)$. Otherwise it need not be considered.

Green–Schwarz Mechanism and Re-derivation of δ_{GS}

Consider a theory with simple gauge group G and an ‘anomalous’ Abelian gauge factor $U(1)_{\mathrm{anom}}$. Under $U(1)_{\mathrm{anom}}$ the dilaton superfield S shifts according to

$$S \;\to\; S + \frac{\mathrm{i}}{2}\delta_{\mathrm{GS}}\, \Lambda(x) \tag{19.50}$$

with Λ denoting the $U(1)_{\mathrm{anom}}$ transformation, i.e. the chiral superfields follow the rule

$$\Phi^{(f)} \;\to\; \mathrm{e}^{-\mathrm{i}\, Q^{(f)}_{\mathrm{anom}}\, \Lambda}\, \Phi^{(f)}\;. \tag{19.51}$$

[5]The Dynkin index is also given by $\ell(\boldsymbol{r}^{(f)}) = T(\boldsymbol{r}^{(f)})$ defined by $T(\boldsymbol{r}^{(f)})\,\delta_{AB} = Tr(T_A\, T_B)$ where T_A is the generator in the $(\boldsymbol{r}^{(f)})$ representation.

The corresponding transformation of the vector superfield V_{anom} is

$$V_{\text{anom}} \;\to\; V_{\text{anom}} + \frac{\mathrm{i}}{2}(\Lambda - \Lambda^{\dagger}) \tag{19.52}$$

with $\mathrm{Re}\Lambda|_{\theta=0} = \alpha$. In what follows, we derive the Green–Schwarz coefficient δ_{GS} from the requirement of invariance of the full action.

The dilaton-dependent part of the Lagrangian density is

$$\begin{aligned}\mathscr{L}_{\text{dilaton}} = &- \int \mathrm{d}^4\theta \, \ln\left(S + S^{\dagger} - \delta_{\text{GS}}\, V_{\text{anom}}\right) \\ &+ \left[\int \mathrm{d}^2\theta \, \frac{S}{4} \, \mathrm{Tr}\, W_{\alpha} W^{\alpha} + \text{h.c.}\right] \\ &+ \text{gravity terms} \,.\end{aligned} \tag{19.53}$$

The first line of this Lagrangian density is already invariant under the combined transformations (19.50) and (19.52). The trace in the second line is supposed to run over all gauge factors, including $U(1)_{\text{anom}}$.

Decomposing the scalar component of the dilaton into a real and an imaginary (or axionic) part,

$$S|_{\theta=0} \;=\; s + \mathrm{i}\, a \,, \tag{19.54}$$

leads to the usual couplings of the axion a

$$\mathscr{L} \;\supset\; -\frac{a}{8} F_{\text{anom}} \widetilde{F}_{\text{anom}} - \frac{a}{8} F^a \widetilde{F}^a + \frac{a}{4} \mathscr{R}\widetilde{\mathscr{R}} \,, \tag{19.55}$$

where F and F_{anom} denote the gauge field strength of G and $U(1)_{\text{anom}}$ respectively.

Hence, under a $U(1)_{\text{anom}}$ transformation with parameter α the axionic Lagrangian density shifts by

$$\Delta\mathscr{L}_{\text{axion}} \;=\; -\frac{\alpha}{16} \delta_{\text{GS}}\, F_{\text{anom}} \widetilde{F}_{\text{anom}} - \frac{\alpha}{16} \delta_{\text{GS}}\, F^a \widetilde{F}^a + \frac{\alpha}{8} \delta_{\text{GS}}\, \mathscr{R}\widetilde{\mathscr{R}} \,. \tag{19.56}$$

The Green–Schwarz term δ_{GS} can now be inferred by demanding that the transformation of the axion a cancels the anomalous variation of the path integral measure. The latter can be absorbed in a change of the Lagrangian density

$$\begin{aligned}\Delta\mathscr{L}_{\text{anomaly}} = &\frac{\alpha}{32\pi^2} F_{\text{anom}} \widetilde{F}_{\text{anom}}\; A_{U(1)^3_{\text{anom}}} \\ &+ \frac{\alpha}{32\pi^2} F^a \widetilde{F}^a\, A_{G-G-U(1)_{\text{anom}}} \\ &- \frac{\alpha}{384\pi^2} \mathscr{R}\widetilde{\mathscr{R}}\, A_{\text{grav}-\text{grav}-U(1)_{\text{anom}}} \,.\end{aligned} \tag{19.57}$$

The coefficients A are the anomaly coefficients given by

$$A_{U(1)^3_{\text{anom}}} = \frac{1}{3}\sum_m (Q^{(m)}_{\text{anom}})^3 \;=\; \frac{1}{3}\,\mathrm{Tr}\,Q^3_{\text{anom}}\;, \tag{19.58a}$$

$$A_{\text{grav}-\text{grav}-U(1)_{\text{anom}}} = \sum_m Q^{(m)}_{\text{anom}} \;=\; \mathrm{Tr}\,Q_{\text{anom}}\;, \tag{19.58b}$$

$$A_{G-G-U(1)_{\text{anom}}} = \sum_{\boldsymbol{r}^{(f)}} \ell(\boldsymbol{r}^{(f)})\,Q^{(f)}_{\text{anom}}\;, \tag{19.58c}$$

where $Q^{(m)}_{\text{anom}}$ denotes the $U(1)_{\text{anom}}$ charge. The first two sums run over all left-handed Weyl fermions while the last sum runs over all irreducible representations $\boldsymbol{r}^{(f)}$ of G and $\ell(\boldsymbol{r}^{(f)})$ is the Dynkin index.

The axion shift allows us to cancel the grav–grav–$U(1)_{\text{anom}}$, $U(1)^3_{\text{anom}}$ and $G-G-U(1)_{\text{anom}}$ anomalies by demanding $\Delta\mathscr{L}_{\text{anomaly}} + \Delta\mathscr{L}_{\text{axion}} = 0$. This fixes the Green–Schwarz constant to be given by

$$2\pi^2\,\delta_{\text{GS}} \;=\; \frac{1}{24}\,\mathrm{Tr}\,Q_{\text{anom}} \;=\; \frac{1}{3}\,\mathrm{Tr}\,Q^3_{\text{anom}} \;=\; A_{G-G-U(1)_{\text{anom}}}\;, \tag{19.59}$$

which is in agreement with the result obtained in a string computation [396].

Discrete Green–Schwarz Mechanism

The Green–Schwarz mechanism also works if we replace $U(1)_{\text{anom}}$ by a discrete $\mathbb{Z}_M$. In this case the parameter α is no longer continuous but $\alpha = \frac{2\pi n}{M}$ with some integer n. Of course, there is no gauge field associated with the $\mathbb{Z}_M$, i.e. Eq. (19.52) does not apply here. The discussion then goes as in the previous subsection. The discrete Green–Schwarz constant is now defined in such a way that under the $\mathbb{Z}_M$ transformation of fields

$$\Phi^{(f)} \;\to\; \mathrm{e}^{-\mathrm{i}\,\frac{2\pi}{M}\,q^{(f)}}\,\Phi^{(f)} \tag{19.60}$$

the dilaton shifts according to

$$S \;\to\; S + \frac{\mathrm{i}}{2}\Delta_{\text{GS}}\;, \tag{19.61}$$

where Δ_{GS} is fixed only modulo η,

$$\pi\,M\,\Delta_{\text{GS}} \;\equiv\; \frac{1}{24}A_{\text{grav}-\text{grav}-\mathbb{Z}_M} \;=\; A_{G-G-\mathbb{Z}_M} \mod \eta\;. \tag{19.62}$$

The anomaly coefficients can be obtained from Eq. (19.58) by replacing the $U(1)_{\mathrm{anom}}$ charges $Q^{(m)}_{\mathrm{anom}}$ by the $\mathbb{Z}_M$ charges $q^{(m)}$. Note that, unlike in the continuous case, the transformation of the axion is only fixed modulo η. In the main body of this chapter we obtain constraints on possible discrete symmetries and charge assignments from the requirement that Eq. (19.62) possesses a solution, i.e. that the $A_{G-G-\mathbb{Z}_M}$ coefficients for different gauge factors G coincide modulo η.

$\mathbb{Z}_M$ and $\mathbb{Z}_M^R$ Anomaly Coefficients

We start by looking at the MSSM amended by ordinary, i.e. non-R, discrete symmetries, where the fermions have the same charges as the superfields $\Phi^{(f)}$ and turn then to the discussion of discrete R symmetries.

Anomaly Coefficients for Non-R $\mathbb{Z}_M$

The anomaly coefficients for discrete non-R $\mathbb{Z}_M$ symmetries are well known [89, 359, 360],

$$A_{G-G-\mathbb{Z}_M} = \sum_{\boldsymbol{r}^{(f)}} \ell(\boldsymbol{r}^{(f)}) \cdot q^{(f)} \;, \tag{19.63a}$$

$$A_{\mathrm{grav}-\mathrm{grav}-\mathbb{Z}_M} = \sum_m q^{(m)} \;. \tag{19.63b}$$

These coefficients can be re-derived in the path integral approach [390] (cf. Section "Appendix: Discrete Anomalies in the Path Integral Approach"). In Eq. (19.63a) we sum over all irreducible representations $\boldsymbol{r}^{(f)}$ of G while in Eq. (19.63b) we sum over all fermions. $\ell(\boldsymbol{r}^{(f)})$ denotes the Dynkin index of the representation $\boldsymbol{r}^{(f)}$. The discrete charges q are integers which are defined modulo M. Moreover, there might be mixed U(1) anomalies if the normalization of the U(1) factors is known. The coefficients are

$$A_{U(1)-U(1)-\mathbb{Z}_M} \;=\; \sum_m q^{(m)} \cdot \left(Q^{(m)}\right)^2 \tag{19.63c}$$

with $Q^{(m)}$ denoting the normalized $U(1)$ charges. We will discuss this coefficient in more detail below.

Traditional anomaly freedom requires that for all anomaly coefficients

$$A \;=\; 0 \mod \eta \,. \tag{19.64}$$

However, discrete anomalies can be canceled by the Green–Schwarz mechanism, in which case one has to demand

$$\begin{aligned} A_{G-G-\mathbb{Z}_M} &= A_{U(1)-U(1)-\mathbb{Z}_M} = \frac{1}{24} A_{\text{grav}-\text{grav}-\mathbb{Z}_M} \\ &= \rho \mod \eta \end{aligned} \tag{19.65}$$

An important comment concerns the mixed U(1)–$\mathbb{Z}_M$ anomaly coefficient (19.63c). Mixed $U(1) - U(1) - \mathbb{Z}_M$ anomalies are mostly ignored as they do not give meaningful constraints unless one knows the normalization of the charges [90, 397]. Typically the sum in Eq. (19.63c) is not invariant under shifting some discrete charges by M. To see this, let us consider the example of hypercharge. We will denote the unnormalized $U(1)_Y$ charge by $Q_Y^{(m)}$. The anomaly coefficient reads

$$A_1 = \sum_m \frac{3}{5} \left(Q_Y^{(m)}\right)^2 q^{(m)} = \rho \mod \eta\,. \tag{19.66}$$

We have the freedom to shift the $\mathbb{Z}_M$ charges by integer multiples of M, i.e. we can define new $\mathbb{Z}_M$ charges $q'^{(m)} = q^{(m)} + k^{(m)} M$ with $k^{(m)} \in \mathbb{Z}$. With the new charges the condition for anomaly freedom is

$$\frac{3}{5} \sum_m \left(Q_Y^{(m)}\right)^2 \left(q^{(m)} + k^{(m)} M\right) = \rho \mod \eta \tag{19.67}$$

$$\Rightarrow \quad A_1 + \frac{3}{5} M \underbrace{\sum_m k^{(m)} \left(Q_Y^{(m)}\right)^2}_{=:n} = \rho \mod \eta\,. \tag{19.68}$$

We can choose the $k^{(m)}$ such that n is an arbitrary integer because, for example, $Q_Y(\overline{E}) = 1$. Hence, we arrive at

$$A_1 = \rho - \frac{3}{5} n M + m \eta\,, \qquad m \in \mathbb{Z}\,. \tag{19.69}$$

This can be rewritten as

$$M \text{ odd} : \quad 5A_1 = 5\rho + (5m - 3n)\, M\,, \tag{19.70}$$

$$M \text{ even} : \quad 5A_1 = 5\rho + (5m - 6n)\, \frac{M}{2}\,. \tag{19.71}$$

Since $5m - 3n$ and $5m - 6n$ are arbitrary integers, we get

$$5A_1 = 5\rho \mod \eta\,. \tag{19.72}$$

Anomaly Coefficients for $\mathbb{Z}_M^R$ Symmetries

Now consider a $\mathbb{Z}_M^R$ symmetry, under which, by convention, the superpotential transforms as

$$\mathscr{W} \;\to\; \mathrm{e}^{2\pi\,\mathrm{i}\,q_{\mathscr{W}}/M}\,\mathscr{W} \tag{19.73}$$

with $q_{\mathscr{W}} = 2$. Accordingly, the superspace coordinates transform as

$$\theta \;\to\; \mathrm{e}^{2\pi\,\mathrm{i}/M}\,\theta\;, \tag{19.74}$$

such that $\mathrm{d}^2\theta$ transforms oppositely to $\mathscr{W}$. Superfields $\Phi^{(f)} = \phi^{(f)} + \sqrt{2}\,\theta\psi^{(f)} + \theta\theta\,F^{(f)}$ follow the law

$$\Phi^{(f)} \;\to\; \mathrm{e}^{2\pi\,\mathrm{i}\,q^{(f)}/M}\,\Phi^{(f)}\;. \tag{19.75}$$

Correspondingly, the fermions transform as

$$\psi^{(f)} \;=\; \mathrm{e}^{2\pi\,\mathrm{i}\,(q^{(f)}-1)/M}\,\psi^{(f)}\;. \tag{19.76}$$

For discrete R symmetries, the anomaly coefficients read (cf. Section "Appendix: Discrete Anomalies in the Path Integral Approach")

$$A_{G-G-\mathbb{Z}_M^R} = \sum_{\boldsymbol{r}^{(f)}} \ell\left(\boldsymbol{r}^{(f)}\right)\cdot(q^{(f)}-1) + \ell(\mathrm{adj}\,G)\;, \tag{19.77a}$$

$$A_{U(1)-U(1)-\mathbb{Z}_M^R} = \sum_m (Q^{(m)})^2\;\cdot(q^{(m)}-1)\;, \tag{19.77b}$$

$$A_{\mathrm{grav}-\mathrm{grav}-\mathbb{Z}_M^R} = -21 + \sum_G \dim(\mathrm{adj}\,G) + \#(U(1)) + \sum_m (q^{(m)}-1)\;. \tag{19.77c}$$

Here $q^{(f)}$ denote the $\mathbb{Z}_M^R$ charges of the superfields, the charges of the corresponding fermions are shifted by one unit, $q_{\psi^{(f)}} = q^{(f)} - 1$. In Eq. (19.77a) $\ell(\mathrm{adj}\,G) = C_2(G)$ represents the contribution from the gauginos, $\#(U(1))$ denotes the number of U(1) gauginos. The first and second term on the right-hand side of Eq. (19.77c) represent the contributions from the gravitino and gauginos. A necessary condition for anomaly cancellation is the universality

$$\begin{aligned} A_{G-G-\mathbb{Z}_M^R} &= A_{U(1)-U(1)-\mathbb{Z}_M^R} = \frac{1}{24}\,A_{\mathrm{grav}-\mathrm{grav}-\mathbb{Z}_M^R} \\ &= \rho \mod \eta\;. \end{aligned} \tag{19.78}$$

ρ is a constant, which is related to the discrete shift (19.61) of the axion via $\rho = \pi\, M\, \Delta_{\mathrm{GS}}$.

Summary of Anomaly Coefficients

The anomaly coefficients are given by

$$A_{G-G-\mathbb{Z}_M^{(R)}} = \sum_{\boldsymbol{r}^{(f)}} \ell(\boldsymbol{r}^{(f)})\, (q^{(f)} - R) + \ell(\mathrm{adj}\, G) \cdot R\,, \tag{19.79a}$$

$$A_{U(1)-U(1)-\mathbb{Z}_M^{(R)}} = \sum_m (Q^{(m)})^2\, (q^{(m)} - R)\,, \tag{19.79b}$$

$$\begin{aligned} A_{\mathrm{grav}-\mathrm{grav}-\mathbb{Z}_M^{(R)}} &= R \cdot \left[-21 + \sum_G \dim(\mathrm{adj}\, G) + \#(U(1))\right] \\ &\quad + \sum_m (q^{(m)} - R)\,, \end{aligned} \tag{19.79c}$$

where we distinguish between discrete non-R ($R = 0$) and R ($R = 1$) symmetries. $\#(U(1))$ denotes the number of U(1) gauginos. As discussed above, the mixed $U(1) - U(1) - \mathbb{Z}_M^{(R)}$ anomaly is only meaningful if one knows the normalization. In general, the coefficient $A_{U(1)-U(1)-\mathbb{Z}_M^{(R)}}$ is not invariant under shifts of the $\mathbb{Z}_M^{(R)}$ charges by integer multiples of M.

Chapter 20
Embedding Orbifold GUTs in the Heterotic String

We have seen that supersymmetric orbifold GUTs retain many of the nice features of 4D GUTs, such as gauge coupling unification and GUT relations for Yukawa couplings. In addition, they replace the complicated GUT breaking and doublet-triplet splitting sectors of the 4D theory by the very elegant mathematics of orbifolding with discrete symmetries. They also open up new mechanisms for supersymmetry breaking. However, on the down side, these orbifold GUT theories are non-renormalizable and therefore an explicit cut-off must be introduced. In this chapter we begin the discussion of embedding supersymmetric orbifold GUTs into string theory. In a string theory, the arbitrary cut-off is replaced by the physical string scale. Moreover, string theory has the benefit of also including quantum gravity.

String theory constructions, which attempt to obtain the MSSM in four dimensions using the analysis of orbifolding and Wilson lines, have been around since the work of Dixon et al. [398, 399] on the $E_8 \times E_8$ heterotic string. MSSM-like models were then constructed by Ibanez et al. [400–402] using the $\mathbb{Z}_3$ orbifold and Wilson lines. The $\mathbb{Z}_3$ orbifold was used because it naturally gave multiples of three families. However, there were inherent difficulties in obtaining realistic models. It was found that R-parity was not, in general, guaranteed; dimension 5 proton decay was typically unsuppressed; Yukawa matrices typically had rank ≤ 3, and many models contain additional exotic states which are chiral under the SM gauge group.[1] Finally, the normalization of hypercharge is typically different than in $SU(5)$ and, moreover, states can exist with strange values of hypercharge [403]. Perhaps a different criterion for searching for string vacua, which can more naturally produce MSSM-like models, is needed.

In recent years, the search for phenomenologically acceptable models has been pursued in Type I and II string models, using the mathematics of D-branes. These

[1] States which are chiral under the Standard Model gauge group can only obtain mass by coupling to the Higgs boson. Thus their mass is necessarily of order the weak scale. Hence there are stringent experimental bounds on such states.

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attempts have also had many problems. This has the lead to the discussion of F-theory models. For a recent review, see [404]. We won't say more about these constructions here. Instead, we focus on heterotic string models and in particular $E_8 \times E_8$ heterotic string models, since these models naturally contain GUT subgroups with the possibility of a realistic spectrum at the perturbative level, i.e.

$$E_8 \supset E_6 \supset SO(10) \supset SU(5) \supset SU(3) \times SU(2) \times U(1). \tag{20.1}$$

Phenomenological Guidelines

In the exploration of the string landscape in [405–410] the following guidelines are used when searching for "realistic" string models.

1. We want to preserve gauge coupling unification.
2. We want to retain low energy SUSY as a solution to the gauge hierarchy problem, i.e. why is $M_Z << M_G$.
3. Quarks and leptons come in complete **16** dimensional representations of $SO(10)$. Thus we want to incorporate this fact directly in the construction.
4. We want to put the Higgs in the **10** dimensional representation of $SO(10)$. Thus quarks and leptons are distinguished from Higgs by their $SO(10)$ quantum numbers.
5. We want to preserve GUT relations for the third family Yukawa couplings.
6. We also want to use the fact that GUTs accommodate a "Natural" See-Saw scale, $\mathcal{O}(M_G)$.

In order to accomplish the above, we use the intuition derived from Orbifold GUT constructions, i.e. we want to embed orbifold GUTs into the heterotic string [383, 411]. We also want to use local GUTs to enforce the family structure [383, 411–414]. We shall find that by imposing orbifold and local GUTs into our string constructions, we are able to find many MSSM-like models.

As a final comment, the string theory analysis discussed here assumes supersymmetric vacua at the string scale. As a consequence there are generically a multitude of moduli. The gauge and Yukawa couplings depend on the values of the moduli vacuum expectation values [VEVs]. In addition vector-like exotics,[2] as well as additional $U(1)$ gauge bosons, can have mass proportional to the moduli VEVs. We will assume arbitrary values for these moduli VEVs, along supersymmetric directions, in order to obtain desirable low energy phenomenology. Of course, at some point supersymmetry must be broken and these moduli must be stabilized. We save this harder problem for a later date. Nevertheless, we can add one more guideline at this point. In the supersymmetric limit, we want the superpotential to have a vanishing VEV. This is so that we can work in flat Minkowski space when considering supergravity. Some of our models naturally have this property.

[2]By definition, a vector-like exotic can obtain mass without breaking any Standard Model gauge symmetry.

Brief Introduction to Heterotic String Theory

SO(32) String in 10D

The classical action for the heterotic string is given by (for books on the subject, see [415, 416]). In this chapter I can only outline the basic ideas.

$$S = -\frac{1}{2\pi}\int d^2\sigma\Big[\sum\nolimits_{\mu=0}^{9}(\partial_\alpha X^\mu\ \partial^\alpha X_\mu - 2i\psi^\mu\partial_+\psi_\mu) \\ - 2i\sum\nolimits_{A=1}^{32}\ \tilde{\lambda}^A\partial_-\tilde{\lambda}^A\Big]. \tag{20.2}$$

The 2D coordinate is $\sigma^\alpha = (\tau,\ \sigma) \equiv (\sigma^0,\ \sigma^1)$ and we also define $\sigma^\pm \equiv \tau \pm \sigma$ with $\partial_\pm \equiv \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. The metric in the conformal gauge is given by $\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. The fields X^μ are bosonic, while the fields $\psi^\mu,\ \tilde{\lambda}^A$ are Majorana-Weyl fermions, i.e. one component fermions.

The fields satisfy Lagrange's equations of motion which are free field equations for closed strings with $\sigma = [0, \pi)$. The solution of the equations of motion are given by left and right movers. The right movers, given by the fields

$$X^\mu(\sigma^-),\quad \psi^\mu(\sigma^-),\quad \mu = 0, \cdots, 9 \tag{20.3}$$

are supersymmetric with the supersymmetry transformations

$$\delta X^\mu = i\epsilon\ \psi^\mu,\quad \delta\psi^\mu = \epsilon\partial_- X^\mu. \tag{20.4}$$

The critical dimension for right movers is 10.

On the other hand the critical dimension for left movers is 26 if all are bosonic. The left movers are given by the fields

$$X^\mu(\sigma^+),\quad \mu = 0, \cdots, 9,\quad \tilde{\lambda}^A(\sigma^+),\quad A = 1, \cdots, 32. \tag{20.5}$$

Note, in 2D two real fermions is equivalent to one real boson. If all $\tilde{\lambda}^A(\sigma^+)$ have the same boundary conditions then there exists an $SO(32)$ symmetry

$$\tilde{\lambda}'^A = O^{AB}\,\tilde{\lambda}^B. \tag{20.6}$$

The bosonic fields satisfy periodic boundary conditions and commutation relations

$$[X^\nu(\sigma,\tau), \dot{X}^\mu(\sigma',\tau)] = i\pi\ \eta^{\mu\nu}\ \delta(\sigma - \sigma'). \tag{20.7}$$

They have the mode expansions

$$X^\mu(\sigma^-)_R = \frac{1}{2}x^\mu + \frac{1}{2}\ell_s^2 p^\mu\,\sigma^- + \frac{i}{2}\ell_s \sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\, e^{-2in\sigma^-} \tag{20.8}$$

and

$$X^\mu(\sigma^+)_L = \frac{1}{2}x^\mu + \frac{1}{2}\ell_s^2 p^\mu\,\sigma^+ + \frac{i}{2}\ell_s \sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu\, e^{-2in\sigma^+}. \tag{20.9}$$

The constants $x^\mu,\ p^\mu$ describe the position and momentum of the string and $\ell_s \sim m_{Pl}^{-1}$ is the string length. $\eta^{\mu\nu} = diag(-1,+1,\cdots,+1)$ is the 10D Lorentz metric. In the light-cone gauge, only the transverse modes of the string are quantized and we have for the right movers

$$[\alpha_m^i,\ \alpha_n^j] = m\,\delta_{m+n,0}\,\eta^{ij}, \quad i,j = 1,\cdots,8 \tag{20.10}$$

and similarly for the left movers

$$[\tilde{\alpha}_m^i,\ \tilde{\alpha}_n^j] = m\,\delta_{m+n,0}\,\eta^{ij}, \quad i,j = 1,\cdots,8. \tag{20.11}$$

For the right moving fermions we have the anti-commutation relations

$$\{\psi^\mu(\sigma^-),\ \psi^\nu(\sigma'^-)\} = \pi\eta^{\mu\nu}\delta(\sigma^- - \sigma'^-). \tag{20.12}$$

The fermions satisfy either periodic (Ramond) or anti-periodic (Neveu-Schwarz) boundary conditions. Both are needed. We have the mode expansions

$$\psi^\mu(\sigma^-) = \sum_{n\in\mathbb{Z}} d_n^\mu\, e^{-2in\sigma^-} \quad \text{R sector} \tag{20.13}$$

$$\sum_{r\in\mathbb{Z}+1/2} b_r^\mu\, e^{-2ir\sigma^-} \quad \text{NS sector.} \tag{20.14}$$

The quantum anti-commutation relations for the transverse mode coefficients is given by

$$\{d_m^i,\ d_n^j\} = \eta^{ij}\delta_{m+n,0} \tag{20.15}$$

$$\{b_r^i,\ b_s^j\} = \eta^{ij}\delta_{r+s,0}. \tag{20.16}$$

Now consider left moving fermions. For the (R) sector we have

$$\tilde{\lambda}^A(\sigma) = \sum_{-\infty}^{+\infty}\tilde{\lambda}_n^A\, e^{-2in\sigma}, \quad n\in\mathbb{Z} \tag{20.17}$$

with anti-commutation relations

$$\{\tilde{\lambda}^A_m,\ \tilde{\lambda}^B_n\} = \delta^{AB}\,\delta_{m+n,0}. \tag{20.18}$$

The anti-periodic (NS) sector satisfies the anti-commutation relations

$$\{\tilde{\lambda}^A_r,\ \tilde{\lambda}^B_s\} = \delta^{AB}\,\delta_{r+s,0}, \quad r, s \in \mathbb{Z} + 1/2. \tag{20.19}$$

In all cases we choose the modes with $m,\ r$ negative as creation operators and $m,\ r$ positive as annihilation operators. We can then define the number operators

$$N_\alpha = \sum_{m=1}^{\infty} \alpha^i_{-m}\alpha^i_m, \quad \tilde{N}_\alpha = \sum_{m=1}^{\infty} \tilde{\alpha}^i_{-m}\tilde{\alpha}^i_m \tag{20.20}$$

$$N_d = \sum_{m=1}^{\infty} m\, d^i_{-m}\, d^i_m, \quad N_b = \sum_{r=1/2}^{\infty} r\, b^i_{-r}\, b^i_r. \tag{20.21}$$

There are similar relations for the anti-commutators of the left moving fermions, $\tilde{\lambda}$. This is fine for the non-zero modes, but what about the zero fermionic modes in the R sector. They satisfy the anti-commutation relations

$$\{d^\mu_0,\ d^\nu_0\} = \eta^{\mu\nu}. \tag{20.22}$$

This is just the Dirac algebra and we can define Γ matrices satisfying

$$\{\Gamma^\mu,\ \Gamma^\nu\} = -2\eta^{\mu\nu} \tag{20.23}$$

by

$$\Gamma^\mu = \sqrt{2}i\, d^\mu_0. \tag{20.24}$$

The ground state of d^μ_0 must be irreducible representations of the Dirac algebra, i.e. spinors of $SO(1, D-1)$. In the light cone gauge the ground state is then an irreducible representation of $SO(8)$. As a consequence, the Neveu-Schwarz sector of the theory gives space-time bosons, while the Ramond sector gives space-time fermions.

The mass operator for the string is given in terms of the total ten momentum carried by the string, i.e. $M^2 = -p_\mu\, p^\mu$. For the right movers we have

$$\frac{M_R^2}{4} = (N_\alpha + N_d) \quad \text{R sector} \tag{20.25}$$

$$(N_\alpha + N_b - \frac{1}{2}) \quad \text{NS sector.}$$

And for the left movers we have

$$\frac{M_L^2}{4} = N_{\tilde{\alpha}} + N_{\tilde{\lambda}_x} - \tilde{a}_x \tag{20.26}$$

where

$$x = \begin{cases} A & NS \\ P & R \end{cases}. \tag{20.27}$$

The constant a_x is a normal ordering constant. We have

$$a_{\text{real boson}} = \frac{1}{24}, \quad a_{\text{real A fermion}} = \frac{1}{48}, \quad a_{\text{real P fermion}} = -\frac{1}{24}. \tag{20.28}$$

Thus we have[3]

$$\tilde{a}_A = 8(\frac{1}{24}) + 32(\frac{1}{48}) = 1 \tag{20.29}$$

and

$$\tilde{a}_P = 8(\frac{1}{24}) + 32(-\frac{1}{24}) = -1. \tag{20.30}$$

We finally obtain

$$\begin{aligned} \frac{M_L^2}{8} = & (N_{\tilde{\alpha}} + N_{\tilde{\lambda}} + 1) \quad \text{R sector} \\ & (N_{\tilde{\alpha}} + N_{\tilde{\lambda}} - 1) \quad \text{NS sector.} \end{aligned} \tag{20.31}$$

For a closed string we have the constraint

$$M_L^2 \,|\text{phys}\rangle \equiv M_R^2 \,|\text{phys}\rangle. \tag{20.32}$$

Now let's work out the massless sector of the theory. The physical states are tensor product states of left and right movers. First consider the massless left moving states, we have

$$\tilde{\alpha}_{-1}^i \,|0\rangle_L, \quad \tilde{\lambda}_{-1/2}^A \,\tilde{\lambda}_{-1/2}^B \,|0\rangle_L. \tag{20.33}$$

The first is a space-time vector, 8_v, and the second is an $SO(32)$ adjoint. Now consider the right moving massless states. We have

$$b_{-1/2}^i \,|0\rangle_R, \quad |8_c\rangle_R = |\dot{a}\rangle_R. \tag{20.34}$$

[3]We could have applied the same analysis to Eq. (20.25).

The first is a space-time vector, 8_v, and the second is a space-time spinor ground state [spinor and bosonic representations of $SO(8)$ (or its $SO(6)$ subgroup) are obtained similar to the analysis of $SO(10)$ in Sect. 5.7]. The weights of 8_v are given by

$$8_v = (\pm 1, 0, 0, 0) \tag{20.35}$$

plus all permutations. While the weights of 8_c are given by the states

$$8_c = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}) \tag{20.36}$$

with an even number of positive signs.

Combining the left and right moving states into a tensor product we have

$$\tilde{\alpha}^i_{-1}\ |0\rangle_L \otimes \begin{cases} b^i_{-1/2}\ |0\rangle_R \\ \quad |8_c\rangle_R \end{cases} . \tag{20.37}$$

In this sector we have 64 bosonic states containing a graviton (ij), dilaton $Tr(ij)$ $(35 + 1)$ and anti-symmetric tensor $[ij]$ (28). We also have 64 fermionic states which decompose into a 56 dimensional gravitino and eight dimensional spinor. Then we also have the states given by

$$\tilde{\lambda}^A_{-1/2}\ \tilde{\lambda}^B_{-1/2}\ |0\rangle_L\ \otimes \begin{cases} b^i_{-1/2}\ |0\rangle_R \\ \quad |8_c\rangle_R \end{cases} . \tag{20.38}$$

These correspond to 496 gauge bosons and gauginos of SUSY $SO(32)$. At higher mass levels the theory gives massive SUSY $SO(32)$ supergravity with states in $Spin(32)/\mathbb{Z}_2$.[4]

$E_8 \times E_8$ String in 10D

Now let's construct heterotic $E_8 \times E_8$. In this case it is convenient to replace the 32 left moving fermions by 16 bosonic degrees of freedom.[5] The action becomes

$$S = -\frac{1}{2\pi}\int d^2\sigma[\sum_{\mu=0}^{9}(\partial_\alpha X^\mu\ \partial^\alpha X_\mu - 2i\psi^\mu\partial_+\psi_\mu) \\ + \sum_{I=1}^{16}\ \partial_+ X^I_L \partial_- X^I_L] \tag{20.39}$$

[4] A GSO projection is needed to guarantee space-time SUSY.

[5] A discussion of the property of bosonization of fermions in $1+1$ dimensions is discussed in more detail in Sect. 20.0.1.

with $X_L^I(\sigma^+)$ only left moving. The mode expansion is then given by

$$X_L^I = x_L^I + P_L^I\,\sigma^+ + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^I\,e^{-2in\sigma^+} \tag{20.40}$$

with the constraint that

$$x_L^I = x_L^I + 2\pi L^I. \tag{20.41}$$

Thus we have compactified the 16 dimensions on a lattice given by

$$2L^I \equiv \sum_{i=1}^{16} n_i\,e_i^I \;\in\; \Gamma^8 \otimes \Gamma^8, \tag{20.42}$$

i.e. where Γ^8 is the root lattice of E_8 and e_i^I are the simple roots of $E_8 \times E_8$. The Cartan matrix is given by

$$g_{ij} \equiv \sum_{I=1}^{16} e_i^I\,e_j^I. \tag{20.43}$$

Since this 16 dimensional space is compactified, the momenta are quantized and we have

$$P^I \;\in\; \Gamma^8 \otimes \Gamma^8 \tag{20.44}$$

or

$$P^I = \{(n_1,\ n_2,\ n_3,\cdots,n_8),(n_1{+}1/2,\ n_2+1/2,\ n_3+1/2,\cdots,n_8+1/2)\};\ \ \sum_{i=1}^{8} n_i \in 2\mathbb{Z}. \tag{20.45}$$

This is for one E_8 and the same is true for the second E_8.[6]

We then have the mass operator for the left movers given by

$$\frac{M_L^2}{4} = N_{\tilde{\alpha}^i} + N_{\tilde{\alpha}^I} + \frac{1}{2}\sum_{I=1}^{16}(P^I)^2 - 1. \tag{20.46}$$

[6]Note, if the coordinates are compactified on a particular lattice, then the momenta are compactified on the dual lattice. However, the lattice Γ^8 is self-dual. Therefore the momenta also reside in the weight space of $E_8 \times E_8$.

The massless modes for left movers are now given by

$$\tilde{\alpha}^{i}_{-1}\,|0\rangle_L,\;\;\tilde{\alpha}^{I}_{-1}\,|0\rangle_L,\;\;|P^2=2\rangle_L. \tag{20.47}$$

The right moving massless modes are the same as before. Thus the gravity sector is unchanged. However the gauge sector is now

$$\left\{\begin{array}{c}\tilde{\alpha}^{I}_{-1}\,|0\rangle_L\\ |P^2=2\rangle_L\end{array}\right\}\otimes\left\{\begin{array}{c}b^{i}_{-1/2}\,|0\rangle_R\\ |8_c\rangle_R\end{array}\right\}. \tag{20.48}$$

This gives the gauge bosons and gauginos for $E_8\times E_8$ where the $\tilde{\alpha}^{I}_{-1}\,|0\rangle_L$ give the gauge bosons in the Cartan sub-algebra and the rest are given by the momentum modes.

$E_8 \times E_8$ String in 4D

Up until now we have found the massless states of the heterotic string in 10D. When we first wrote down the heterotic string action we assumed that the conformal fields in the light-cone gauge, X^i, propagated in the transverse target space with metric δ_{ij}. Now we want to compactify six dimensions on a 6D spatial torus. In general the action for the bosonic modes of the string in a non-flat background is given by

$$S=-\frac{1}{2\pi}\int d^2\sigma\;[G_{ij}\partial_\alpha X^i\partial^\alpha X^j+\partial_\alpha X^I\partial^\alpha X^I+\epsilon^{\alpha\beta}A^I_i\partial_\alpha X^i\partial_\beta X^I+\epsilon^{\alpha\beta}B_{ij}\partial_\alpha X^i\partial_\beta X^j] \tag{20.49}$$

where G_{ij} is the metric, B_{ij} is an anti-symmetric tensor field and A_i is a Wilson line.[7] Here we will take $B_{ij}=0$ and $G_{ij},\ A_i=$ constants. Let us first consider the metric $G_{ab}=R_aR_b\alpha_a\cdot\alpha_b$ where α_a are the lattice vectors defining the torus. Later we will introduce the Wilson lines. Starting with N = 1 SUSY in 10D where a spinor is eight dimensional (if we don't break SUSY) we obtain N = 4 SUSY in 4D. But we want N = 1 SUSY in 4D, so we will orbifold the torus to break N = 4 SUSY to N = 1, and also get a chiral theory. Consider the simplest case, and the first one studied historically, a $\mathbb{Z}_3$ orbifold of the torus, $T^6=(T^2)^3$.

First let us just study the orbifold of T^2. Consider the two torus defined by modding out translations along the direction of the simple roots of an $SU(3)$ root

[7] Note if the backgrounds, $G,\ B$ depend on X, then the model is equivalent to the non-linear sigma model. If one requires that the theory remain conformally invariant, i.e. the relevant beta functions vanish, one obtains at one loop the equations of motion for the fields $G,\ B$ which are the Einstein equations of gravity coupled to the anti-symmetric tensor field, see Chap. 3.4 of [415].

lattice with roots

$$\alpha_1 = (\sqrt{2}, 0), \quad \alpha_2 = (-\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}). \tag{20.50}$$

In the light-cone gauge with the dynamical boson field, $X^i, \quad i = 1, \cdots, 8$, we choose to compactify the directions $i = 1, \cdots, 6$ on T^6 and let $i = 7, 8$ correspond to the transverse directions of space-time which are not compact. Consider now just $X^i, \ \ i = 1, 2$ compactified on an $SU(3)$ torus. We have

$$x^i = x^i + 2\pi L^i \ \text{ with } \ L^i = \sum_{a=1}^{2} n_a \, \alpha_a^i \, R, \quad n_a \in \mathbb{Z} \tag{20.51}$$

where

$$x^i = x_L^i + x_R^i. \tag{20.52}$$

The field $X^i(\sigma, \tau)$ also satisfies the closed string boundary conditions

$$X^i(\sigma + \pi, \tau) = X^i(\sigma, \tau) + 2\pi L^i. \tag{20.53}$$

With these boundary conditions the field $X^i(\sigma, \tau)$ has the following mode expansion

$$X^i(\sigma, \tau) = x^i + P^i\tau + 2L^i\sigma + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^i \, e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^i \, e^{-2in(\tau+\sigma)}]. \tag{20.54}$$

The momenta, P^i, take values in the dual lattice and we have

$$P^i = \sum_{a=1}^{2} \frac{m_a \, \alpha_a^{i*}}{R}, \quad m_a \in \mathbb{Z} \tag{20.55}$$

with

$$\alpha_1^* = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}), \quad \alpha_2^* = (0, \sqrt{\frac{2}{3}}) \tag{20.56}$$

satisfying

$$\alpha_a^* \cdot \alpha_b \equiv \delta_{ab}. \tag{20.57}$$

We can also define $X^i_L,\ X^i_R$ by

$$X^i_L(\sigma^+) = x^i_L + p^i_L\,\sigma^+ + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\,\tilde{\alpha}^i_n\,e^{-2in\,\sigma^+} \tag{20.58}$$

$$X^i_R(\sigma^-) = x^i_R + p^i_R\,\sigma^- + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\,\alpha^i_n\,e^{-2in\,\sigma^-} \tag{20.59}$$

with

$$p^i_L = \frac{1}{2}(P^i + 2L^i), \quad p^i_R = \frac{1}{2}(P^i - 2L^i), \quad x^i = x^i_L + x^i_R. \tag{20.60}$$

It will be convenient to define complex coordinate, $Z(\sigma,\tau)$. We have

$$Z = \frac{X^1 + iX^2}{\sqrt{2}} = z + p\tau + 2L\sigma + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}[\alpha_n\,e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n\,e^{-2in(\tau+\sigma)}]. \tag{20.61}$$

The complex creation and annihilation operators satisfy the commutation relations

$$[\alpha^\dagger_m, \alpha_n] = m\delta_{n+m,0}, \quad [\tilde{\alpha}^\dagger_m, \tilde{\alpha}_n] = m\delta_{n+m,0}, \quad [z^\dagger, p] = i. \tag{20.62}$$

Similarly for the right moving fermions we define the complex fermion field

$$\Psi(\sigma^-) = \frac{1}{\sqrt{2}}(\psi^1(\sigma^-) + i\psi^2(\sigma^-)). \tag{20.63}$$

We have the mode expansions

$$\Psi(\sigma^-) = \sum_{n\in\mathbb{Z}} d_n\,e^{-2in\sigma^-} \quad \text{R sector} \tag{20.64}$$

$$\sum_{r\in\mathbb{Z}+1/2} b_r\,e^{-2ir\sigma^-} \quad \text{NS sector} \tag{20.65}$$

the quantum anti-commutation relations given by

$$\{d^\dagger_m,\ d_n\} = \delta_{m+n,0} \tag{20.66}$$

$$\{b^\dagger_r,\ b_s\} = \delta_{r+s,0}. \tag{20.67}$$

We shall also compactify the other two torii on $SU(3)$ lattices. For the three torii in the complex plane we have $Z^i = \frac{X^{2i-1} + i\,X^{2i}}{\sqrt{2}}$ for $i = 1, 2, 3$ and correspondingly for the right moving fermions we have $\Psi^i(\sigma^-),\ i = 1, 2, 3$ where $Z^4,\ \Psi^4$ are in the transverse direction of our non-compactified space.

The mass operator for right movers is given by

$$\frac{M_R^2}{4} = (N_B + N_F) + \frac{1}{2}(\mathbf{p_R})^2 - (a_B + a_F) = (N_B + N_F) + \frac{1}{2}(p_R^i)^2 - \begin{cases} 0 & [R] \\ 1/2 & [NS] \end{cases} \tag{20.68}$$

and the mass operator for left movers is

$$\frac{M_L^2}{4} = \tilde{N}_B + \frac{1}{2}(\mathbf{p_L})^2 + \frac{1}{2}(\mathbf{P})^2 - \tilde{a}_B = \tilde{N}_B + \frac{1}{2}(\mathbf{p_L})^2 + \frac{1}{2}(\mathbf{P})^2 - 1. \tag{20.69}$$

The number operators are given by

$$\begin{aligned}
N_B &= \sum_{n=1}^{\infty}[: \alpha_{-n}^{\dagger 4}\alpha_n^4 : + : \alpha_{-n}^4\alpha_n^{\dagger 4} : \\
&\quad + \sum_{i=1,2,3} : \alpha_{-n}^{\dagger i}\alpha_n^i : + \sum_{i=1,2,3} : \alpha_{-n}^i\alpha_n^{\dagger i} :] \\
\tilde{N}_B &= \sum_{n=1}^{\infty}[: \tilde{\alpha}_{-n}^{\dagger 4}\tilde{\alpha}_n^4 : + : \tilde{\alpha}_{-n}^4\tilde{\alpha}_n^{\dagger 4} : \\
&\quad + \sum_{i=1,2,3} : \tilde{\alpha}_{-n}^{\dagger i}\tilde{\alpha}_n^i : + \sum_{i=1,2,3} : \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^{\dagger i} : + \sum_{I=1}^{16} : \alpha_{-n}^I\alpha_n^I :] \\
N_F[R] &= \sum_{n=1}^{\infty} n\,[: d_{-n}^{\dagger 4}d_n^4 : + : d_{-n}^4 d_n^{\dagger 4} : \\
&\quad + \sum_{i=1,2,3} : d_{-n}^{\dagger i}d_n^i : + \sum_{i=1,2,3} : d_{-n}^i d_n^{\dagger i} :] \\
N_F[NS] &= \sum_{r=71/2}^{\infty} r\,[: b_{-r}^{\dagger 4}b_r^4 : + : b_{-r}^4 b_r^{\dagger 4} : \\
&\quad + \sum_{i=1,2,3} : b_{-r}^{\dagger i}b_r^i : + \sum_{i=1,2,3} : b_{-r}^i b_r^{\dagger i} :].
\end{aligned}$$

The normal ordering constants for real bosons and fermions are given by

$$a_B = \tilde{a}_B = \frac{1}{24}, \quad a_F = -\frac{1}{24} + \frac{1}{4}(v_0 - \frac{1}{2})^2, \quad \text{with} \quad v_0 = \begin{cases} \frac{1}{2} & [R] \\ 0 & [NS] \end{cases}. \tag{20.70}$$

Each $SU(3)$ lattice is invariant under a $120°$ rotation which defines a $\mathbb{Z}_3$ symmetry. Consider just one torus, we have for example $Z^{1'} = e^{2\pi i/3}\, Z^1$. Defining the orbifold by $T^2/\mathbb{Z}_3$ we find three fixed points in this torus given by the solution

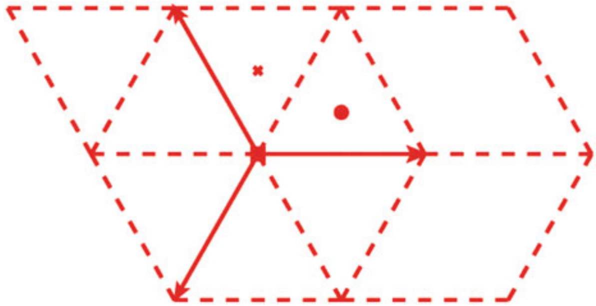

Fig. 20.1 The $SU(3)$ orbifold has three fixed points given by the x, *filled circle* and *square* at the origin

to the equation

$$e^{2\pi i/3}\, z_f^1 = z_f^1 + L, \quad L \in SU(3) \text{ root lattice} \tag{20.71}$$

(see Fig. 20.1).

Now we are ready to compactify six dimensions on the orbifold $SU(3)^3/\mathbb{Z}_3$. The three two torii are labeled by Z^i, $i = 1, 2, 3$. And we orbifold by the rotation

$$Z'^i = e^{2\pi i v_i}\, Z^i \tag{20.72}$$

where the twist vector, $\mathbf{v}$, satisfies $3v_i \in \mathbb{Z}$. This is equivalent to a rotation θ given by

$$\theta = \exp[2\pi i(v_1 J_{12} + v_2 J_{34} + v_3 J_{56})] \equiv \exp[2\pi i(\mathbf{v} \cdot \mathbf{J})] \tag{20.73}$$

where J_{ij} correspond to the diagonal Cartan generators of $SO(6)$ in the ij plane and $\mathbf{J} = (J_{12}, J_{34}, J_{56})$. In order to preserve one supersymmetry in the 4D theory, one requires that $\sum_{i=1}^3 v_i \in 2\mathbb{Z}$ or equivalently that the $\theta \in SU(3)$.[8]

The gravitinos arise from the right movers with quantum numbers, Eq. (20.36). Under the twist we have

$$\theta|\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\rangle = e^{2\pi i(v_1 J_{12} + v_2 J_{34} + v_3 J_{56})}|\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\rangle. \tag{20.74}$$

The rotation acts on the first, second and third spin 1/2 state in the tensor product. The fourth spin 1/2 state is untouched. The first three spin states transform as a **4** or

[8]Rotations by 4π on fermions is equivalent to the identity and one requires that the rotation generators in $SU(3)$ are orthogonal to the $U(1)$ in the decomposition of $SO(6) \to SU(3) \times U(1)$. The $U(1)$ generator is given by $J_{12} + J_{34} + J_{56}$ which gives $\sum_{i=1}^3 v_i = 0$ (see Sect. 5.7 for the definition of vector and spinor representations in $SO(10)$, since the mathematics is identical for any $SO(2N)$ group).

$\bar{\mathbf{4}}$ representation of $SO(6)$. Under the $SU(3)$ subgroup of $SO(6)$ the $\mathbf{4}$ decomposes into a $\mathbf{3}+\mathbf{1}$ and since the rotation is an element of $SU(3)$ it only acts on the triplet; leaving one supersymmetry unbroken. This breaks the N = 4 supersymmetry in 4D down to N = 1 and we are able to obtain a chiral gauge theory. Note, if $v_3 = 0$, then the rotation is in $SU(2)$ and we remain with N = 2 SUSY in 4D.

Our six torus is a direct product of three versions of Fig. 20.1. The string degrees of freedom include closed strings which are free to propagate anywhere in the torus. This includes modes which wind around the torus as in Eq. (20.54). The mass operator for these modes is unchanged. However, only states which are invariant under the orbifolding are left in the spectrum. To understand this better we need to resort to a bit more mathematics.

We defined the 6D orbifold by $\Omega = T^6/\mathscr{P}$ where $\mathscr{P}$ is a point group such as the $\mathbb{Z}_3$ under consideration. In general $\Omega = R^6/S$ where R^6 is the 6D flat Riemann space and S is called the space group of transformations. In most cases these two definitions are equivalent [417]. The space group of transformations is a discrete version of rotations and translations. We have an element $g \in S$ given by $g = (\theta, a)$ such that when acting on a point in $x \in R^6$ we have

$$g : x \to \theta x + a \tag{20.75}$$

where $\theta \in \mathscr{P}$ and a is a translation. Given two elements $g_1,\ g_2 \in S$ we obtain the product rule

$$g_2 \circ g_1 = (\theta_2\, \theta_1, \theta_2\, a_1 + a_2). \tag{20.76}$$

Physical states on the orbifold are invariant under the action of S. *However, it turns out that, in order to avoid anomalies, we must also embed the space group into the gauge group, $E_8 \times E_8$.*[9] The action of the space group on the gauge group is a homomorphism of the action on the 6D internal space. We define the action

$$G \equiv (\Theta,\ V) \in \mathscr{G} \approx S \tag{20.77}$$

where positions on the $E_8 \times E_8$ gauge lattice transform as follows

$$X_L^I \to \Theta_J^I\, X_L^J + \pi V^I. \tag{20.78}$$

The point group is represented either as an automorphism, Θ, or a shift, V. For example, when acting with a rotation on the fermionized bosons

$$\Psi_L^I(\sigma^+) =: e^{2iX_L^I(\sigma^+)} : \tag{20.79}$$

[9]In the heterotic string, this statement is a consequence of modular invariance, which is a subject which goes way beyond the scope of these lectures. Modular invariance is the reason why string theory is finite, i.e. no loop divergences. See vol. 2 of [415] or Chap. 6 of [416].

we have

$$\Psi'^{I}_{L}(\sigma^+) = e^{2\pi i V^I}\ \Psi^I_L(\sigma^+) \tag{20.80}$$

which implies

$$X^I_L \to X^I_L + \pi V^I. \tag{20.81}$$

Physical string states are necessarily invariant under the product,

$$S \otimes \mathcal{G}. \tag{20.82}$$

The action of $\mathcal{G}$, S on the left and right movers is given by

$$\begin{aligned} G|P\rangle_L &= e^{2i(\pi\mathbf{V})\cdot\hat{\mathbf{P}}}|\Theta P\rangle_L \\ S(\alpha^i_{-n}\ |0\rangle_{R,L}) &= \theta\ (\alpha^i_{-n}\ |0\rangle_{R,L}) \\ G(\tilde{\alpha}^I_{-n}\ |0\rangle_L) &= \Theta^I{}_J(\tilde{\alpha}^J_{-n}\ |0\rangle_L) \\ S\ (b^i_{-r}\ |0\rangle_R) &= (\theta^{-1})^i{}_j\ (b^j_{-r}\ |0\rangle_R) \end{aligned} \tag{20.83}$$

where $\hat{P}$ is the momentum operator. We have a closed string satisfying

$$X^I_L(\sigma+\pi,\tau) = GX^I_L(\sigma,\tau) + \Gamma^I = \Theta^I{}_J\, X^J_L(\sigma,\tau) + \pi V^I. \tag{20.84}$$

A rotation on the fermion degrees of freedom is represented by a gauge twist $\mathbf{V}$ in the boson lattice. Thus in our case we take $\Theta^I{}_J = \delta^I{}_J$ then a solution is given by

$$X^I_L(\sigma^+) = x^I_L + (P^I + V^I)\ \sigma^+ + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\ \tilde{\alpha}^I_n\ e^{-2in\sigma^+}. \tag{20.85}$$

Before discussing the massless spectrum of the orbifolded string, there is one further condition we need to discuss. Not all shift vectors, $\mathbf{V}$, are allowed by modular invariance [398, 399, 418, 419]. In fact for a $\mathbb{Z}_N$ twist, the shift vectors must satisfy the constraint

$$N(\mathbf{V}^2 - \mathbf{v}^2) = 0\ \in 2\mathbb{Z}. \tag{20.86}$$

Consider now the transformation of the physical states. The gauge bosons are given by

$$\tilde{\alpha}^I_{-1}\ |0\rangle_L \otimes b^i_{-1/2}\ |0\rangle_R, \quad |P^I\rangle_L \otimes b^i_{-1/2}\ |0\rangle_R; \ \text{ with }\ i = 1,\cdots,8,\ \ P^2 = 2. \tag{20.87}$$

The invariant states satisfy

$$\mathbf{P} \cdot \mathbf{V} - \mathbf{r_i} \cdot \mathbf{v} \in \mathbb{Z} \tag{20.88}$$

where $\mathbf{r_i}, \quad i = 1, 2, 3, 4$ are the eigenvalues of the rotation generators. We have plus or minus the vectors

$$\mathbf{r}_1 = (1, 0, 0, 0), \quad \mathbf{r}_2 = (0, 1, 0, 0), \quad \mathbf{r}_3 = (0, 0, 1, 0), \quad \mathbf{r}_4 = (0, 0, 0, 1). \tag{20.89}$$

The vector $\mathbf{r}_4$ is the non-compact direction and thus gives the transverse components of the gauge bosons. Thus for the gauge symmetry we have $\mathbf{P}\cdot\mathbf{V} \in \mathbb{Z}$ and it is broken to a subgroup, $H \in E_8 \times E_8$. For example, let us take the twist vector on the 6D torus $SU(3)^3$ as

$$\mathbf{v} = \frac{1}{3}(1, -1, 0, 0). \tag{20.90}$$

We obtain massless gauge bosons in H for $i = 7, 8$ and a chiral multiplet in the adjoint representation of H for $i = 5, 6$. All of these states describe closed strings moving in the 8D bulk and they are massless irreducible representations of N $= 2$ SUSY in 4D. This is due to the fact that we left one torus untouched by the twist.

Now consider the twisted sectors. Corresponding to each fixed point of the orbifold there are so-called twisted sector states (see Fig. 20.2). The mode expansion in the k^{th} twisted sector with $kv_i \neq 0$ is given by

$$Z^i(\sigma + \pi, \tau) = e^{2\pi i k v_i}\, Z^i(\sigma, \tau) \tag{20.91}$$

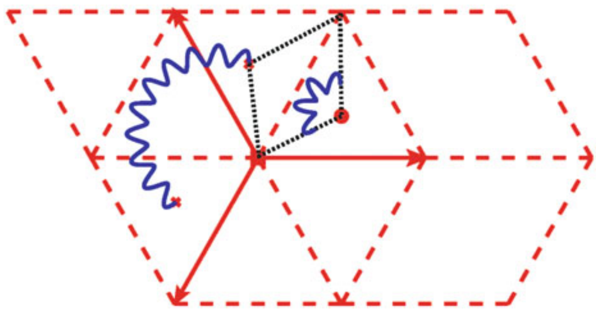

Fig. 20.2 Two examples of closed strings on the twisted $SU(3)$ orbifold. Inside the *dotted lines* defines the fundamental domain of the $\mathbb{Z}_3$ orbifold. It looks like a Hamentashen (three sided cookie). There is a winding mode on the *left* and a twisted sector state on the *right*

with solution

$$Z^i(\sigma,\tau) = z^i{}_{f,k} + \frac{i}{2}\sum_{n\neq 0}\{\frac{\alpha^i{}_{(n+kv_i)}}{n+kv_i}\,e^{-2i(n+kv_i)(\tau-\sigma)} + \frac{\tilde{\alpha}^i{}_{(n-kv_i)}}{n-kv_i}\,e^{-2i(n-kv_i)(\tau+\sigma)}\} \tag{20.92}$$

where $z^i{}_{f,k}$ are the fixed points in the kth twisted sector, i.e. applying the twist k times. Note, all twisted sectors must be summed over. For the right moving fermions we have

$$\psi^i_R(\tau-\sigma-\pi) = e^{2\pi i(v_0-1/2+kv_i)}\,\psi^i_R(\tau-\sigma) \tag{20.93}$$

with solution

$$\psi^i_R(\tau-\sigma) = \begin{cases} \sum_{n\in\mathbb{Z}} d^i_{(n+kv_i)}\,e^{-2i(n+kv_i)(\tau-\sigma)} & [R] \\ \sum_{r\in\mathbb{Z}+1/2} b^i_{(r+kv_i)}\,e^{-2i(r+kv_i)(\tau-\sigma)} & [NS] \end{cases}. \tag{20.94}$$

The creation and annihilation operators satisfy the commutation relations

$$[\alpha^i_{m+kv_i},\alpha^{\dagger j}_{n-kv_j}] = (m+kv_i)\delta^{i,j}\delta_{n+m,0}, \quad [\tilde{\alpha}^i_{m+kv_i},\tilde{\alpha}^{\dagger j}_{n-kv_j}] = (m+kv_i)\delta^{i,j}\delta_{n+m,0}. \tag{20.95}$$

Similarly for the right moving fermions we have the quantum anti-commutation relations given by

$$\{d^i_{m+kv_i},\ d^{\dagger j}_{n-kv_j}\} = \delta^{i,j}\delta_{m+n,0} \tag{20.96}$$

$$\{b^i_{r+kv_i},\ b^{\dagger j}_{s-kv_j}\} = \delta^{i,j}\delta_{r+s,0}. \tag{20.97}$$

The mass operators for the twisted sectors ($k\neq 0$) are given by

$$\frac{M_R^2}{4} = (N_B(k\mathbf{v}) + N_F(k\mathbf{v})) - (a_B(k\mathbf{v}) + a_F(k\mathbf{v})) \tag{20.98}$$

$$\frac{M_L^2}{4} = \tilde{N}_B(k\mathbf{v}) + \frac{1}{2}(\mathbf{P}+k\mathbf{V})^2 - \tilde{a}_B(k\mathbf{v}). \tag{20.99}$$

with

$$\begin{aligned} N_B(k\mathbf{v}) = \sum_{n=1}^{\infty}[&:\alpha^{\dagger 4}_{-n}\alpha^4_n: + :\alpha^4_{-n}\alpha^{\dagger 4}_n: \\ &+ \sum_{i=1,2,3}:\alpha^{\dagger i}_{-n-kv_i}\alpha^i_{n+kv_i}: + \sum_{i=1,2,3}:\alpha^i_{-n+kv_i}\alpha^{\dagger i}_{n-kv_i}:] \end{aligned}$$

$$\begin{aligned}
\tilde{N}_B(k\mathbf{v}) &= \sum_{n=1}^{\infty}[: \tilde{\alpha}^{\dagger 4}_{-n}\tilde{\alpha}^{4}_{n} : + : \tilde{\alpha}^{4}_{-n}\tilde{\alpha}^{\dagger 4}_{n} : \\
&\quad + \sum_{i=1,2,3} : \tilde{\alpha}^{\dagger i}_{-n-kv_i}\tilde{\alpha}^{i}_{n+kv_i} : + \sum_{i=1,2,3} : \tilde{\alpha}^{i}_{-n+kv_i}\tilde{\alpha}^{\dagger i}_{n-kv_i} : + \sum_{I=1}^{16} : \alpha^{I}_{-n}\alpha^{I}_{n} :] \\
N_F(k\mathbf{v})[R] &= \sum_{n=1}^{\infty}[n \ : d^{\dagger 4}_{-n} d^{4}_{n} : + n \ : d^{4}_{-n} d^{\dagger 4}_{n} : \\
&\quad + \sum_{i=1,2,3} (n + kv_i) \ : d^{\dagger i}_{-n-kv_i} d^{i}_{n+kv_i} : + \sum_{i=1,2,3} (n - kv_i) \ : d^{i}_{-n+kv_i} d^{\dagger i}_{n-kv_i} :] \\
N_F(k\mathbf{v})[NS] &= \sum_{r=1/2}^{\infty}[r \ : b^{\dagger 4}_{-r} b^{4}_{r} : + r \ : b^{4}_{-r} b^{\dagger 4}_{r} : \\
&\quad + \sum_{i=1,2,3} (r + kv_i) \ : b^{\dagger i}_{-r-kv_i} b^{i}_{r+kv_i} : + \sum_{i=1,2,3} (r - kv_i) \ : b^{i}_{-r+kv_i} b^{\dagger i}_{r-kv_i} :].
\end{aligned}$$

The normal ordering constants for complex bosons and fermions are given by[10]

$$\begin{aligned}
a_B(kv_i) = \tilde{a}_B(kv_i) = \frac{1}{12} - \frac{1}{2}\,|kv_i|(1 - |kv_i|), \quad a_F(kv_i) = \frac{1}{24} - \frac{1}{2}(kv_i - v_0)^2, \\
\text{with} \quad v_0 = \begin{cases} \frac{1}{2} & [R] \\ 0 & [NS] \end{cases}.
\end{aligned} \tag{20.102}$$

The normal ordering constant for the left movers is then given by

$$\tilde{a}_B(k\mathbf{v}) = 1 - \frac{1}{2}\sum_{i=1}^{3}|kv_i|(1 - |kv_i|). \tag{20.103}$$

[10] In general, the normal ordering coefficient for real bosons is given by

$$a_B(\eta) \equiv -\frac{1}{2}\sum_{n=0}^{\infty}(n + \eta) = \frac{1}{24} - \frac{1}{4}\,\eta(1 - \eta) \equiv -\frac{1}{48} + \frac{1}{4}(\eta - \frac{1}{2})^2. \tag{20.100}$$

For real fermions we have

$$a_F(\eta) = -a_B(\eta). \tag{20.101}$$

20.0.1 Bosonization of Fermions

A complex free fermion in $1+1$ dimensions can be represented by a real boson.[11] We have the formula

$$\Psi(\sigma^{\pm}) =: e^{2i\phi(\sigma^{\pm})} : . \tag{20.104}$$

If $\phi(\sigma^{\pm})$ has the mode expansion

$$\phi(\sigma^{\pm}) = \phi_0 + p_0\,\sigma^{\pm} + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-2in\sigma^{\pm}}, \tag{20.105}$$

then we have

$$\Psi(\sigma^{\pm}) = \exp(-2\sum_{n<0}\frac{1}{n}e^{-2in\sigma^{\pm}}\alpha_n)\; e^{2i\phi_0}\; e^{2i\sigma^{\pm}(p_0+1/2)}\;\exp(-2\sum_{n>0}\frac{1}{n}e^{-2in\sigma^{\pm}}\alpha_n). \tag{20.106}$$

Note, if

$$\Psi(\sigma^{\pm}+\pi) = -\Psi(\sigma^{\pm}), \tag{20.107}$$

then the momenta, $p_0 \in \mathbb{Z}$, while if

$$\Psi(\sigma^{\pm}+\pi) = \Psi(\sigma^{\pm}), \tag{20.108}$$

then the momenta, $p_0 \in \mathbb{Z} + 1/2$.

Bosonization is a statement that the Green's functions of the two sides of the equality are identical. We have

$$\langle\Psi(\sigma^{\pm})\,\Psi^{\dagger}(\sigma'^{\pm})\rangle = \frac{1}{\sigma^{\pm}-\sigma'^{\pm}} = \langle : e^{2i\phi(\sigma^{\pm})} :: e^{-2i\phi(\sigma'^{\pm})} :\rangle. \tag{20.109}$$

The action for fermions is replaced by the corresponding action for bosons. In particular, for right moving fermions we have

$$\frac{i}{\pi}\int d^2\sigma\;\Psi(\sigma^-)\partial_+\Psi(\sigma^-) \Longrightarrow -\frac{1}{2\pi}\int d^2\sigma\;\partial_+\phi(\sigma^-)\partial_-\phi(\sigma^-). \tag{20.110}$$

[11] See, for example, Chap. 3.2.4, [415].

Thus when we bosonize the right moving Ramond or Neveu-Schwarz fermions the mass operator for right movers becomes (in the untwisted sector, see Eq. (20.68))

$$\frac{M_R^2}{4} = \hat{N}_B + \frac{1}{2}(\mathbf{p_R})^2 + \frac{1}{2}(\mathbf{r})^2 - 1/2 \tag{20.111}$$

where $\mathbf{r}$ is the momentum vector of the bosonized fermion, $\mathbf{r} = (n_1, n_2, n_3, n_4)$ for NS fermions and $\mathbf{r} = (n_1 + 1/2, n_2 + 1/2, n_3 + 1/2, n_4 + 1/2)$ for R fermions with $n_i \in \mathbb{Z}$. $\hat{N}_B$ now includes the bosonized fermion oscillator modes. In the k^{th} twisted sector [see Eq. (20.112)] we have

$$\frac{M_R^2}{4} = \hat{N}_B(kv) + \frac{1}{2}(\mathbf{r} + k\mathbf{v})^2 - a_B(k\mathbf{v}) \tag{20.112}$$

with $a_B(k\mathbf{v}) = \frac{1}{2} - \frac{1}{2}\sum_{i=1}^{3} |kv^i|(1 - |kv^i|)$.

$E_8 \times E_8$ Heterotic String Compactified on $(SU(3))^3/\mathbb{Z}_3$ Orbifold with "Standard Embedding"

As a simple example of a string orbifold, consider $(SU(3))^3/\mathbb{Z}_3$ with shift vector

$$\mathbf{v} = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, 0) \tag{20.113}$$

and gauge embedding

$$\mathbf{V} = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, 0, 0, 0, 0, 0)(0, 0, 0, 0, 0, 0, 0, 0) \equiv (\frac{2}{3}, (-\frac{1}{3})^2, 0^5)(0^8). \tag{20.114}$$

$\mathbf{V}$ is only non-zero on the first E_8. This is the so-called "standard embedding."

Let's consider the massless sector of the theory. Consider first the untwisted sector with mass operators given in [see Eqs. (20.69), (20.111)]. The massless states are

$$\tilde{\alpha}^I_{-1}|0\rangle_L \otimes |r^i\rangle_R, \quad |P^I\rangle_L \otimes |r^i\rangle_R, \quad \tilde{\alpha}^i_{-1}|0\rangle_L \otimes |r^i\rangle_R \text{ with } \mathbf{P}^2 = \mathbf{r}^2 = 2 \tag{20.115}$$

subject to the constraint

$$\mathbf{P} \cdot \mathbf{V} - \mathbf{r} \cdot \mathbf{v} = \mathbb{Z}. \tag{20.116}$$

The momentum vector $\mathbf{r}$ takes on values[12]

$$\mathbf{r} = (\underline{\pm 1, 0, 0, 0}) \;\text{ and }\; (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}) \tag{20.117}$$

corresponding to space-time bosons and fermions, respectively. Since the theory is supersymmetric, let's just identify the massless space-time bosonic modes. The non-compactified direction is given by $\pm\mathbf{r_4} = (0, 0, 0, \pm 1)$. This gives us the two helicity modes of gauge bosons which satisfy the constraint $\mathbf{P} \cdot \mathbf{V} = \mathbb{Z}$. The first 16 states are in the Cartan subalgebra of $E_8 \times E_8$. All the non-Cartan generators for the second E_8 satisfy the constraint. Hence this E_8 is unbroken. For the first E_8, the only non-Cartan generators which satisfy the constraint are given by

$$\pm\,(0, 1, -1, 0^5), \qquad \pm(1, \underline{-1, 0}, 0^5), \qquad (0^3, \underline{\pm 1, \pm 1, 0, 0, 0}),$$
$$\pm\tfrac{1}{2}(+1, +1, +1, [+1, +1, +1, +1, +1]) \tag{20.118}$$

where the square bracket represents even sign flips. The first two vectors correspond to the non-Cartan generators of $SU(3)$, while the latter two correspond to the non-Cartan generators of $SO(10)$ and a $\mathbf{16} + \overline{\mathbf{16}}$ which, together with 6 $U(1)$ generators, make a $\mathbf{78}$, i.e. the adjoint representation of E_6. Thus the first E_8 is broken to $SU(3) \times E_6$.

Now consider the matter fields in the untwisted sector satisfying

$$\mathbf{P} \cdot \mathbf{V} - \mathbf{r} \cdot \mathbf{v} = \mathbb{Z} \tag{20.119}$$

with $\pm\mathbf{r_i},\ i = 1, 2, 3$ and $\mathbf{r_1} = (1, 0, 0, 0),\ \mathbf{r_2} = (0, 1, 0, 0),\ \mathbf{r_3} = (0, 0, 1, 0)$. We have $\mathbf{r_i} \cdot \mathbf{v} = \frac{2}{3},\ \ mod(\mathbb{Z})$ for $i = 1, 2, 3$. Thus we need $\mathbf{P} \cdot \mathbf{V} = \frac{2}{3},\ \ mod(\mathbb{Z})$. We have the states satisfying this constraint given by

$$\left\{\begin{array}{c} \frac{1}{2}(+1, -1, -1,\ [+1, +1, +1, +1, +1]) \\ \frac{1}{2}(-1, \underline{-1, +1},\ [+1, +1, +1, +1, +1]) \\ (0, \underline{1, 0},\ \underline{\pm 1, 0, 0, 0, 0}) \\ (1, 0, 0,\ \underline{\pm 1, 0, 0, 0, 0}) \\ (0, -1, -1,\ 0, 0, 0, 0, 0) \\ (-1, \underline{-1, 0}, 0, 0, 0, 0, 0) \end{array}\right. \tag{20.120}$$

The states in the first two rows transform as a $\mathbf{3}$ of $SU(3)$ and a $\mathbf{16}$ of $SO(10)$, while the next two transform as a $\mathbf{3}$ of $SU(3)$ and a $\mathbf{10}$ of $SO(10)$ and the last two transform as a $\mathbf{3}$ of $SU(3)$ and singlets of $SO(10)$. Together they form the $(\mathbf{3}, \mathbf{27})$ of

[12] Underline means include all permutations.

$SU(3) \times E_6$. The states which satisfy $\mathbf{r_i} \cdot \mathbf{v} = \frac{1}{3}, \quad mod(\mathbb{Z})$ transform as a $(\bar{\mathbf{3}}, \overline{\mathbf{27}})$ of $SU(3) \times E_6$. Recall, string theory is a first quantization, so the $(\mathbf{3}, \mathbf{27})$ and their charge conjugates are the states in one complex scalar field with these quantum numbers. Thus counting $\mathbf{r_i},\ i = 1, 2, 3$ we find a multiple of $3(\mathbf{3}, \mathbf{27})$ states of complex scalar fields and their supersymmetric partners. Finally the states of the form

$$\tilde{\alpha}^i_{-1}|0\rangle_L \otimes |r_j\rangle_R \tag{20.121}$$

give the transverse modes of a spin two graviton, complex dilaton and additional moduli along with their supersymmetric partners.

Now consider the massless twisted sector states. The mass operators are given in Eqs. (20.99), (20.103) and (20.112). Each torus contains three fixed points, thus each twisted sector $k = 1, 2$ contains 27 fixed points. The twisted sector states with $k = 2$ are the conjugates of the twisted sector states with $k = 1$, so we will only consider $k = 1$. We have

$$\frac{M_L^2}{4} = \tilde{N}_B(v) + \frac{1}{2}(\mathbf{P} + \mathbf{V})^2 - \frac{2}{3} \tag{20.122}$$

and

$$\frac{M_R^2}{4} = \hat{N}_B(v) + \frac{1}{2}(\mathbf{r} + \mathbf{v})^2 - \frac{1}{6}. \tag{20.123}$$

Without oscillator modes the left moving states satisfy the massless condition have $(\mathbf{P} + \mathbf{V})^2 = \frac{4}{3}$ and similarly the right movers satisfy the condition $(\mathbf{r} + \mathbf{v})^2 = \frac{1}{3}$, i.e. these modes are given by

$$|\mathbf{P} + \mathbf{V}\rangle_L \otimes |\mathbf{r} + \mathbf{v}\rangle_R. \tag{20.124}$$

These conditions are satisfied by the left momenta

$$\mathbf{P} \supset (0, 1, 1,\ 0^5),\quad (-1, 0, 0,\ \underline{\pm 1, 0^4}),\quad \frac{1}{2}(-1, +1, +1,\ [+1, +1, +1, +1, -1]). \tag{20.125}$$

There are only two right momenta satisfying the massless condition. They are

$$-\mathbf{r_1},\quad \frac{1}{2}(-1, 1, 1, -1). \tag{20.126}$$

Together they give the particle states in the chiral multiplet, $(\mathbf{1}, \mathbf{27})$.[13] In addition there are massless left moving oscillator modes given by[14]

$$\tilde{\alpha}^{\dagger 1}_{-1/3}|\mathbf{P}+\mathbf{V}\rangle_L, \quad \tilde{\alpha}^{\dagger 2}_{-1/3}|\mathbf{P}+\mathbf{V}\rangle_L, \quad \tilde{\alpha}^{\dagger 3}_{-1/3}|\mathbf{P}+\mathbf{V}\rangle_L \tag{20.129}$$

with $\frac{1}{2}(\mathbf{P}+\mathbf{V})^2 = \frac{1}{3}$. The left momenta satisfying this condition are given by

$$\mathbf{P} \supset (0^8), \quad (-1, \underline{+1, 0}, \ 0^5). \tag{20.130}$$

Hence at each fixed point we have $3(\bar{3}, 1)$ chiral multiplets. In order for these states to remain in the spectrum they should be invariant under the orbifold $\times$ gauge twist. We have

$$e^{2\pi i[(\mathbf{P}+\mathbf{V})\cdot\mathbf{V}-(\mathbf{r}+\mathbf{v})\cdot\mathbf{v}]}\theta \ |physical\rangle = |physical\rangle \tag{20.131}$$

where θ acts on the oscillator modes [Eqs. (20.73) and (20.83)]. In this case all the massless twisted sector states are invariant under the twists. Hence in the twisted sector we find the states

$$81(\bar{\mathbf{3}}, \mathbf{1}) + 27(\mathbf{1}, \mathbf{27}). \tag{20.132}$$

20.0.2 Wilson Lines

The holonomy associated with a Wilson line was defined as

$$T_i = \exp[i \oint A^I_i \ dx \cdot H^I] \tag{20.133}$$

---

[13]Note, the vectors in Eq. (20.125) satisfy

$$\mathbf{P}+\mathbf{V} = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \ 0^5), \quad (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \ \underline{\pm 1, 0^4}), \quad \frac{1}{2}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ [+1, +1, +1, +1, -1]). \tag{20.127}$$

Thus the vectors sit at the origin in the shifted $SU(3)$ lattice. Note, the simple roots of $SU(3)$ are given by the unit vector $\alpha_1 = \mathbf{e_1} - \mathbf{e_2}$ and $\alpha_2 = \mathbf{e_2} - \mathbf{e_3}$ with $\mathbf{e_i} \cdot \mathbf{e_j} = \delta_{ij}$.

[14]Note, the vectors in Eq. (20.130) satisfy

$$\mathbf{P}+\mathbf{V} = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, \ 0^5), \quad (-\frac{1}{3}, \underline{\frac{2}{3}, -\frac{1}{3}}, \ 0^5). \tag{20.128}$$

Thus the vectors are in the $\bar{\mathbf{3}}$ of the shifted $SU(3)$ lattice.

where H^I is one of the Cartan generators and $\oint A_i^I \, dx \equiv 2\pi \, a_i^I$ or

$$T_i = \exp[2\pi i \, a_i^I \cdot H^I]. \tag{20.134}$$

When acting on a state with gauge momentum **P** we have

$$T_i \, |\mathbf{P}\rangle_L = \exp(2\pi i \, a_i \cdot \mathbf{P})|\mathbf{P}\rangle_L. \tag{20.135}$$

When do we pick up this phase? When we translate in the direction of the Wilson line. For example, the orbifold boundary conditions were given by

$$X^i(\sigma + \pi, \tau) = (\theta X)^i(\sigma, \tau) + 2\pi L^i. \tag{20.136}$$

In the gauge sector, we then have the additional shift

$$X^I(\sigma^+ + \pi) = X^I(\sigma^+) + \pi V^I + \pi \sum_{i=1}^{6} m_i \, a_i^I \tag{20.137}$$

with $m_i \in \mathbb{Z}$. Note the integer value of m_i depends on the number and direction of the shifts necessary to return to the fixed point after a twist. Therefore twisted sector states are distinguished by the fixed point they sit on. As a consequence, the gauge momenta get translated by

$$P^I \to P^I + V^I + \sum_{i=1}^{6} \, m_i \, a_i^I. \tag{20.138}$$

The Wilson lines associated with discrete orbifold parities are also discrete. For example in the case of $SU(3)/\mathbb{Z}_3$, the action of the twist on the simple roots,

$$\theta\alpha_1 = \alpha_2 \tag{20.139}$$

implies that

$$a_1 = a_2. \tag{20.140}$$

Also

$$\theta\alpha_2 = -\alpha_1 - \alpha_2 \tag{20.141}$$

implies that

$$a_2 = -a_1 - a_2 = -2a_2 \quad \text{or} \quad 3a_2 = 3a_1 = 0 \tag{20.142}$$

up to gauge lattice shifts.

In the untwisted sector the gauge momentum satisfy

$$\mathbf{P} \cdot \mathbf{V} \in \mathbb{Z}, \quad \mathbf{P} \cdot \mathbf{X_{n_f}} \in \mathbb{Z} \tag{20.143}$$

where $\mathbf{X_{n_f}} = \mathbf{V} + \sum m_i \; \mathbf{a_i}$ and the index n_f specifies the number of translations associated with the particular fixed point. Hence the Wilson line further breaks the gauge symmetry. In the twisted sectors the mass shell condition is given by

$$\frac{M_L^2}{4} = \tilde{N}_B(v) + \frac{1}{2}(\mathbf{P} + k\mathbf{X_{n_f}})^2 - \tilde{a}_B(v). \tag{20.144}$$

Moreover, twisted sector states may have different local gauge symmetries. Finally, modular invariance requires

$$N[\mathbf{v}^2 - (\mathbf{X_{n_f}})^2] \in 2\mathbb{Z}. \tag{20.145}$$

There are also additional constraints on the allowed Wilson lines which we consider in later sections.

20.1 Embedding Orbifold GUTs in the Heterotic String

This section is based on the results of Kobayashi et al. [383, 411]. We have already discussed many of the properties of orbifold GUT field theory models. To summarize, these GUT models utilize properties of higher-dimensional field theories, and have some advantages over conventional 4D GUTs. Recall, for example, GUT symmetry breaking can be accomplished by an orbifold parity, instead of by a complicated Higgs sector. The doublet-triplet splitting problem, which plagues conventional GUTs, can also be solved by assigning appropriate orbifold parities to the doublet and triplet Higgs bosons. Note, however, that like all field theoretical models in higher dimensions, these GUT models are not renormalizable quantum field theories. They can only make sense as low-energy effective theories of some more fundamental theory with better ultra-violet (UV) behavior. Our string models provide exactly such UV completions.

To make the connections between string and field theoretical models more concrete, we consider the example of a 5D orbifold GUT model with bulk gauge group E_6 studied in Chap. 16. In this model, the extra dimension is taken to be an orbicircle $S^1/\mathbb{Z}_2$ and the 4D effective theory has a Pati-Salam (PS) symmetry, $SU_{4C} \otimes SU_{2L} \otimes SU_{2R}$. We shall construct the UV completion of this model in the heterotic string.

The technical apparatus we adopt to build string models is the simplest Abelian symmetric orbifold compactification of the heterotic string. More specifically, we consider a non-prime-order $\mathbb{Z}_6$ orbifold (or equivalently, $\mathbb{Z}_2 \times \mathbb{Z}_3$) model with the orbifold twist vector $\mathbf{v}_6 = \frac{1}{6}(1, 2, -3)$. To achieve three chiral PS families at low energies, we also introduce several (in fact, two) discrete Wilson lines [400].[15]

It is obvious that the third compactified complex dimension in the $\mathbb{Z}_6$ model has a $\mathbb{Z}_2$ symmetry, hence it can consistently be taken to be the root lattice of the SO_4 Lie algebra. The string models are effectively 5D when the length of one of the SO_4 simple roots is large compared to the string scale, while all other dimensions are kept comparable to the string scale (i.e. the geometry of the compactified space is equivalent to that of the orbifold GUTs, $S^1/\mathbb{Z}_2$). In this limit, the $\mathbb{Z}_6$ heterotic models are similar to the orbifold GUT models in the following respects:

- After acting with the $\mathbb{Z}_3$ orbifold only on the two small torii we obtain an effective 6D space, $M_4 \times T^2$. The remaining torus has one large and one small radius. Thus it is effectively a 5D theory on $M_4 \times S^1$. In addition the resulting theory has an $N = 2$ supersymmetry.[16]
- The $N = 2$ supersymmetry is then broken to that of $N = 1$ in 4D by the $\mathbb{Z}_2$ orbifold twist and the "bulk" gauge group is broken to two different regular subgroups at the two inequivalent fixed points by a degree-2 non-trivial gauge embedding and Wilson line. Note, these correspond to so-called 'local GUTs' in the literature [410, 427]. The surviving gauge group in the 4D effective theory is the intersection of groups at the fixed points. It is the PS group in our models. We have an E_6 symmetry in the 5D bulk which is broken at the two fixed points to SO_{10} and $SU_6 \times SU_2$ respectively.
- Untwisted-sector and twisted-sector states that are not localized on the $\mathbb{Z}_2$ fixed points of the SO_4 lattice can be identified with the "bulk" states of the orbifold GUT. Interpretation of the Kaluza-Klein (KK) towers of the bulk gauge and matter fields agree in the string-based and orbifold GUT models.
- Twisted-sector states that are localized on the $\mathbb{Z}_2$ fixed points of the SO_4 lattice have no field theoretical counterparts, although they can correctly be identified with the "brane" states of the orbifold GUT. In the orbifold GUT models, these states are only constrained by the requirement of (chiral) anomaly cancellation.

Of course, string theoretical models are more intricate than the corresponding field theoretic orbifold GUT models. They need to satisfy more stringent consistency conditions and thus they are physically more constrained. We find it is highly non-trivial (or impossible) to implement all the features of the orbifold GUTs. For example, we cannot arbitrarily place the three families of quarks and leptons in

[15] Prime-order orbifold models (such as the $\mathbb{Z}_3$ orbifold models) with Wilson lines [401, 420–424] and non-prime-order orbifold models without Wilson lines [425, 426] have been extensively studied in the literature. Non-prime-order orbifold models with Wilson lines, on the other hand, possess a number of complications, and to our knowledge they have not been studied to the same extent. Our work can be regarded as the first serious attempt at constructing three-family models from non-prime-order orbifolds.

[16] By $N = 2$ supersymmetry in 5 or 6D, we mean the minimal number of supersymmetries in these dimensions, (i.e. the fermions satisfy the pseudo-reality condition). It reduces to $N = 2$ in 4D by dimensional reduction and is sometimes called $N = 1$ supersymmetry in the literature.

the bulk or on either brane. Moreover, the very act of obtaining three families, along with their respective locations, is fixed by the requirement that the gauge embeddings and Wilson lines have to satisfy the modular invariance conditions. In addition, we cannot utilize the orbifold projections to remove all the $(\mathbf{6}, \mathbf{1}, \mathbf{1})$ color-triplet states as in the SO_{10} orbifold GUTs and at the same time obtain three families. We also find many massless states in unconventional representations under the SM gauge group. These exotic states are commonplace in almost all known three-family models. Whether these models can give rise to satisfactory phenomenology needs more detailed knowledge of the low-energy effective actions. A class of models with MSSM symmetry in 4D is discussed next. These models come very close to having a realistic spectrum with non-trivial Yukawa matrices and Majorana neutrino masses.

To summarize, in Chap. 16 we introduced a novel orbifold GUT in 5D with gauge group E_6. It is a novel 5D model with many nice phenomenological features. Now in Sect. 20.3 we discuss the heterotic string construction of the model. Using this model as a guide we compare the heterotic string construction with generic orbifold GUT models by restricting the compactified space to a specific type (which is referred to as the orbifold GUT limit). We show the equivalence between the matter states (in the untwisted and some twisted sectors) in string-based models and the bulk states in orbifold GUTs, as well as their KK excitations. We interpret orbifold parities (for the bulk states) in the orbifold GUTs in string theory language, and explain why the gauge embeddings and Wilson lines cannot project away all the $(\mathbf{6}, \mathbf{1}, \mathbf{1})$ color-triplet states. These states may be needed to break the PS group to that of the SM, as in the field theoretical model of Chap. 16. In addition the theory has no chiral exotics. Finally, we show that the model has the very nice feature of a non-Abelian D_4 family symmetry which can constrain fermion mass matrices.

20.2 Heterotic String on a $\mathbb{Z}_6$ Orbifold

We compactify the theory on 6D torus defined by the space group action of translations on a factorizable Lie algebra lattice $G_2 \oplus SU(3) \oplus SO(4)$ (see Fig. 20.3). Then we mod out by the $\mathbb{Z}_6$ action on the three complex compactified coordinates

Fig. 20.3 $G_2 \oplus SU(3) \oplus SO(4)$ lattice. Note, we have taken five directions with string scale length ℓ_s and one with length $2\pi R \gg \ell_s$. This will enable the analogy of an effective 5D orbifold field theory

given by $Z^i \to e^{2\pi i \mathbf{r}_i \cdot \mathbf{v}_6} Z^i$, $i = 1, 2, 3$, where

$$\mathbf{v}_6 = \frac{1}{6}(1, 2, -3, 0) \tag{20.146}$$

is the twist vector, and $\mathbf{r}_1 = (1, 0, 0, 0)$, $\mathbf{r}_2 = (0, 1, 0, 0)$, $\mathbf{r}_3 = (0, 0, 1, 0)$.[17] For simplicity and definiteness, we also take the compactified space to be a factorizable Lie algebra lattice $G_2 \oplus SU(3) \oplus SO(4)$ (see Fig. 20.3).

The $\mathbb{Z}_6$ orbifold is equivalent to a $\mathbb{Z}_2 \times \mathbb{Z}_3$ orbifold, where the two twist vectors are $\mathbf{v}_2 = 3\mathbf{v}_6 = \frac{1}{2}(1, 0, -1, 0)$ and $\mathbf{v}_3 = 2\mathbf{v}_6 = \frac{1}{3}(1, -1, 0, 0)$. The $\mathbb{Z}_2$ and $\mathbb{Z}_3$ sub-orbifold twists have the $SU(3)$ and $SO(4)$ planes as their fixed torii. In Abelian symmetric orbifolds, gauge embeddings of the point group elements and lattice translations are realized by shifts of the momentum vectors, $\mathbf{P}$, in the $E_8 \times E_8$ root lattice[18] [401, 420, 423–425], i.e., $\mathbf{P} \to \mathbf{P} + k\mathbf{V} + l\mathbf{W}$, where k, l are some integers, and $\mathbf{V}$ and $\mathbf{W}$ are the gauge twists and Wilson lines [400]. These embeddings are subject to modular invariance requirements [398, 399, 418]. The Wilson lines are also required to be consistent with the action of the point group. In the $\mathbb{Z}_6$ model, there are at most three consistent Wilson lines [428, 429], one of degree 3 ($\mathbf{W}_3$), along the $SU(3)$ lattice, and two of degree 2 ($\mathbf{W}_2$, $\mathbf{W}_2'$), along the $SO(4)$ lattice.

The $\mathbb{Z}_6$ model has three untwisted sectors (U_i, $i = 1, 2, 3$) and five twisted sectors (T_i, $i = 1, 2, \cdots, 5$). (The T_k and T_{6-k} sectors are CPT conjugates of each other.) The twisted sectors split further into sub-sectors when discrete Wilson lines are present. In the $SU(3)$ and $SO(4)$ directions, we can label these sub-sectors by their winding numbers, $n_3 = 0, 1, 2$ and n_2, $n_2' = 0, 1$, respectively. In the G_2 direction, where both the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ sub-orbifold twists act, the situation is more complicated. There are four $\mathbb{Z}_2$ fixed points in the G_2 plane. Not all of them are invariant under the $\mathbb{Z}_3$ twist, in fact three of them are transformed into each other. Thus for the T_3 twisted-sector states one needs to find linear combinations of these fixed-point states such that they have definite eigenvalues, $\gamma = 1$ (with multiplicity 2), $e^{i2\pi/3}$, or $e^{i4\pi/3}$, under the orbifold twist [428–430] (see Fig. 20.4). Similarly, for the $T_{2,4}$ twisted-sector states, $\gamma = 1$ (with multiplicity 2) and -1 (the fixed points of the $T_{2,4}$ twisted sectors in the G_2 torus are shown in Fig. 20.5). The T_1 twisted-sector states have only one fixed point in the G_2 plane, thus $\gamma = 1$ (see Fig. 20.6). The eigenvalues γ provide another piece of information to differentiate twisted sub-sectors.

Massless states in 4D string models consist of those momentum vectors $\mathbf{P}$ and $\mathbf{r}$ ($\mathbf{r}$ are in the $SO(8)$ weight lattice) which satisfy the following mass-shell equations

[17] Together with $\mathbf{r}_4 = (0, 0, 0, 1)$, they form the set of positive weights of the $\mathbf{8}_v$ representation of the $SO(8)$, the little group in 10d. $\pm\mathbf{r}_4$ represent the two uncompactified dimensions in the light-cone gauge. Their space-time fermionic partners have weights $\mathbf{r} = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ with even numbers of positive signs; they are in the $\mathbf{8}_s$ representation of $SO(8)$. In this notation, the fourth component of $\mathbf{v}_6$ is zero.

[18] The E_8 root lattice is given by the set of states $\mathbf{P} = \{n_1, n_2, \cdots, n_8\}$, $\{n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, \cdots, n_8 + \frac{1}{2}\}$ satisfying $n_i \in \mathbb{Z}$, $\sum_{i=1}^{8} n_i = 2\mathbb{Z}$.

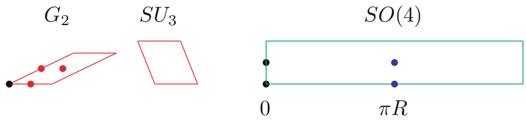

Fig. 20.4 $G_2 \oplus SU(3) \oplus SO(4)$ lattice with $\mathbb{Z}_2$ fixed points. The T_3 twisted sector states sit at these fixed points. The fixed point at the origin and the symmetric linear combination of the *red* (*grey*) fixed points in the G_2 torus have $\gamma = 1$

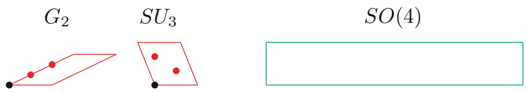

Fig. 20.5 $G_2 \oplus SU(3) \oplus SO(4)$ lattice with $\mathbb{Z}_3$ fixed points for the T_2 twisted sector. The fixed point at the origin and the symmetric linear combination of the *red* (*grey*) fixed points in the G_2 torus have $\gamma = 1$

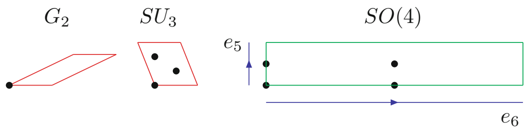

Fig. 20.6 $G_2 \oplus SU(3) \oplus SO(4)$ lattice with $\mathbb{Z}_6$ fixed points. The T_1 twisted sector states sit at these fixed points

[398, 399, 401, 420, 423–425],

$$\frac{M_R^2}{4} = N_R^k + \frac{1}{2}|\mathbf{r} + k\mathbf{v}|^2 - a_R^k = 0\,, \tag{20.147}$$

$$\frac{M_L^2}{4} = N_L^k + \frac{1}{2}|\mathbf{P} + k\mathbf{X}|^2 - a_L^k = 0\,, \tag{20.148}$$

where N_R^k and N_L^k are (fractional) numbers of the right- and left-moving (bosonic) oscillators, $\mathbf{X} = \mathbf{V} + n_3\mathbf{W}_3 + n_2\mathbf{W}_2 + n_2'\mathbf{W}_2'$, and a_R^k, a_L^k are the normal ordering constants,

$$a_R^k = \frac{1}{2} - \frac{1}{2}\sum_{i=1}^{3} |\widehat{kv_i}| \left(1 - |\widehat{kv_i}|\right)\,,$$

$$a_L^k = 1 - \frac{1}{2}\sum_{i=1}^{3} |\widehat{kv_i}| \left(1 - |\widehat{kv_i}|\right)\,, \tag{20.149}$$

with $\widehat{kv_i} = \mathrm{mod}(kv_i, 1)$.

These states are subject to a generalized Gliozzi-Scherk-Olive (GSO) projection $\mathscr{P} = \frac{1}{6}\sum_{\ell=0}^{5} \Delta^{\ell}$ [401, 420, 423–425]. For the simple case of the k-th twisted sector ($k = 0$ for the untwisted sectors) with no Wilson lines ($n_3 = n_2 = n_2' = 0$) we have

$$\Delta = \gamma\phi \exp\left\{ i\pi\left[(2\mathbf{P} + k\mathbf{X}) \cdot \mathbf{X} - (2\mathbf{r} + k\mathbf{v}) \cdot \mathbf{v} \right]\right\}, \tag{20.150}$$

where ϕ are phases from bosonic oscillators. However, in the $\mathbb{Z}_6$ model, the GSO projector must be modified for the untwisted-sector and $T_{2,4}$, T_3 twisted-sector states in the presence of Wilson lines [383]. The Wilson lines split each twisted sector into sub-sectors and there must be additional projections with respect to these sub-sectors. This modification in the projector gives the following projection conditions,

$$\mathbf{P} \cdot \mathbf{V} - \mathbf{r}_i \cdot \mathbf{v} = \mathbb{Z} \quad (i = 1, 2, 3), \quad \mathbf{P} \cdot \mathbf{W}_3,\ \mathbf{P} \cdot \mathbf{W}_2,\ \mathbf{P} \cdot \mathbf{W}_2' = \mathbb{Z}, \tag{20.151}$$

for the untwisted-sector states, and

$$T_{2,4} : \mathbf{P} \cdot \mathbf{W}_2,\ \mathbf{P} \cdot \mathbf{W}_2' = \mathbb{Z}, \qquad T_3 : \mathbf{P} \cdot \mathbf{W}_3 = \mathbb{Z}, \tag{20.152}$$

for the $T_{2,3,4}$ sector states (since twists of these sectors have fixed torii). There is no additional condition for the T_1 sector states.

20.3 Heterotic String Construction of Effective Orbifold GUTs

We construct a three-family $\mathbb{Z}_6$ orbifold model with two Wilson lines. The details are given in Sect. 20.5. We have obtained the complete spectra of massless states (plus KK excitations for these models in certain limits). As we now show, this model is the string equivalent to the orbifold GUT in Chap. 16.

There are three Kähler class moduli ($\mathscr{T}_{1,2,3}$), whose real parts parameterize the sizes of the three tori, and one complex structure modulus ($\mathscr{U}_3$), which parameterizes the shape of the third torus. Explicitly, $\mathrm{Re}\mathscr{T}_3 = 2RR' \sin\phi$, and $\mathscr{U}_3 = \frac{R}{R'}\,\mathrm{e}^{\mathrm{i}\phi}$, where R, R' are the lengths of the two axes of the SO_4-lattice and ϕ their relative angle. These moduli are arbitrary parameters. One may make the length of one axis (along which one puts the degree-2 Wilson line, $\mathbf{W}_2$), say R, large compared to the string length scale while keeping all other dimensions small. In this limit (for length scales larger than the string scale but smaller than the radius R), the low energy theory is effectively five dimensional.[19] The SO_4 lattice, on which

[19] It should be obvious that our construction can be generalized to 6D models, simply by taking both R and R' large compared to the string length scale. These models are related to 6D orbifold GUTs compactified on $\mathrm{T}^2/\mathbb{Z}_2$.

only the $\mathbb{Z}_2$ sub-orbifold twist acts, has four fixed points. With only one degree-2 Wilson line, the fixed points split into two inequivalent classes, labelled by the winding number $n_2 = 0, 1$. Thus in our setup the fifth dimension is equivalent to the orbicircle $S^1/\mathbb{Z}_2$ where each of the two fixed points has a degree-2 degeneracy.

Note that we find it convenient to reinterpret the $\mathbb{Z}_6$ model in terms of the equivalent $\mathbb{Z}_2 \times \mathbb{Z}_3$ orbifold (where the $\mathbb{Z}_2$ ($\mathbb{Z}_3$) sub-orbifold twist acts on the G_2 and SO_4 (G_2 and SU_3) sub-lattices). This point of view is more useful for our comparisons with the orbifold GUT in Chap. 16. Labelling a twisted sector in the $\mathbb{Z}_6$ model by T_k where $k = 1, 2, \cdots, 5$ and in the $\mathbb{Z}_2 \times \mathbb{Z}_3$ model by $T_{(k,l)}$ where $k = 0, 1$ and $l = 0, 1, 2$, then the correspondence between the twisted sectors in the $\mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$ orbifolds is the following:

$\mathbb{Z}_6$ orbifold	T_1	T_2	T_3	T_4	T_5
$\mathbb{Z}_2 \times \mathbb{Z}_3$ orbifold	$T_{(1,2)}$	$T_{(0,1)}$	$T_{(1,0)}$	$T_{(0,2)}$	$T_{(1,1)}$

. (20.153)

The $T_{2,4}$ sectors, which will shortly be identified with the bulk states in the language of orbifold GUTs, have $k = 0,\ l = 1, 2$; therefore they are untwisted by the $\mathbb{Z}_2$ twist.

20.4 PS Model from the $\mathbb{Z}_6$ Orbifold Compactification

We now examine the model defined by the shift vector $\mathbf{V_6}$ and Wilson lines $\mathbf{W_3},\ \mathbf{W_2}$ given in Sect. 20.5. Consider first the model with only the $\mathbb{Z}_3$ sub-orbifolding being imposed (i.e., with twist vector $\mathbf{v}_3 = 2\mathbf{v}_6$, gauge twist $\mathbf{V}_3 = 2\mathbf{V}_6$ and a degree-3 Wilson line $\mathbf{W}_3$, where $\mathbf{v}_6$, $\mathbf{V}_6$ and $\mathbf{W}_3$ are given in Eqs. (20.146), (20.170) and (20.171)), we find a 6D $N = 2$ model with observable-sector gauge group E_6 (modulo abelian factors). Matter fields of the observable sector consist of effective 4D $N = 2$ hypermultiplets[20] in the following representations (see Fig. 20.7),

$$U\,\text{sectors}:\ \mathbf{27} + \overline{\mathbf{27}}, \qquad T\,\text{sectors}:\ 3 \times (\mathbf{27} + \overline{\mathbf{27}})\,. \tag{20.154}$$

The remaining $\mathbb{Z}_2$ twist acts as a space reversal on the third compactified complex dimension, $Z_3 \to -Z_3$. The $\mathbb{Z}_3$ models have two gravitini with the SO_8 momentum vectors, $\mathbf{r} = \frac{1}{2}(1, 1, 1, 1)$ and $\frac{1}{2}(1, -1, -1, 1)$, in the Ramond sector of the right-moving superstring. Only one of them, $\mathbf{r} = \frac{1}{2}(1, 1, 1, 1)$, satisfies the $\mathbb{Z}_2$ projection, $\mathbf{r} \cdot \mathbf{v}_2 = \mathbb{Z}$. Hence the $N = 2$ supersymmetry is broken to that of $N = 1$ in 4D.

Gauge symmetry breaking induced by the $\mathbb{Z}_2$ orbifolding is as follows. The twist vector $\mathbf{v}_2$ is embedded in the gauge degrees of freedom in two different ways, with gauge twists $\mathbf{V}_2$ and $\mathbf{V}_2' = \mathbf{V}_2 + \mathbf{W}_2$ where $\mathbf{V}_2 = 3\mathbf{V}_6$ and $\mathbf{W}_2$ is given in Eq. (20.171). E_6 generators in the Cartan-Weyl basis are transformed under the $\mathbb{Z}_2$

[20] In terms of 4D $N = 1$ chiral superfields.

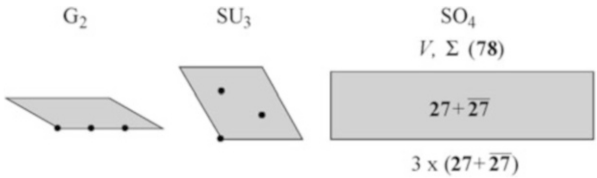

Fig. 20.7 $G_2 \oplus SU_3 \oplus SO_4$ lattice with $\mathbb{Z}_3$ fixed points. The fields $V,\ \Sigma,$ and $\mathbf{27}(\in U_1) + \overline{\mathbf{27}}(\in U_2)$ are bulk states from the untwisted sectors. On the other hand, $3 \times (\mathbf{27} + \overline{\mathbf{27}})$ are "bulk" states located on the $T_{(0,1)}/T_{(0,2)}$ twisted sector (G_2, SU_3) fixed points. *Reprinted from Nuclear Physics B 704, T. Kobayashi, S. Raby and R.J. Zhang, "Searching for realistic 4d string models with a Pati–Salam symmetry. Orbifold grand unified theories from heterotic string compactification on a $\mathbb{Z}_6$ orbifold," Page 15, Copyright (2005), with permission from Elsevier*

action as $E_{\mathbf{P}} \to e^{2\pi i \mathbf{P}\cdot\mathbf{V}_2} E_{\mathbf{P}}$ and $E_{\mathbf{P}} \to e^{2\pi i \mathbf{P}\cdot\mathbf{V}_2'} E_{\mathbf{P}}$, thus the linearly-realized gauge groups consist of roots satisfying $\mathbf{P}\cdot\mathbf{V}_2$ and $\mathbf{P}\cdot\mathbf{V}_2' = \mathbb{Z}$ respectively. The pattern of symmetry breaking in the observable sector can be summarized as follows:

$$\begin{tikzcd} & SO_{10} \arrow[dr] & \\ E_6 \arrow[ur] \arrow[dr] & & PS \\ & SU_6 \times SU_{2R} \arrow[ur] & \end{tikzcd} \tag{20.155}$$

At the final step we have the complete $\mathbb{Z}_6$ model with two discrete Wilson lines being imposed simultaneously; this gives the PS symmetry group in the 4D effective theory.

In these two inequivalent implementations of the $\mathbb{Z}_2$ twist the non-trivial matter fields of SO_{10} and $SU_6 \times SU_{2R}$ are:

Sectors	SO_{10}	$SU_6 \times SU_{2R}$
U_1	$\mathbf{16}$	$(\mathbf{15}, \mathbf{1})$
U_2	$\mathbf{10}$	$(\mathbf{6}, \mathbf{2})$
U_3	$\mathbf{16} + \overline{\mathbf{16}}$	$(\mathbf{20}, \mathbf{2})$
$T_{(0,1)}$	$2 \times \mathbf{16}_+ + \mathbf{10}_-$	$2(\overline{\mathbf{6}}, \mathbf{2})_+ + (\mathbf{15}, \mathbf{1})_-$
$T_{(0,2)}$	$\overline{\mathbf{16}}_- + 2 \times \mathbf{10}_+$	$(\mathbf{6}, \mathbf{2})_- + 2(\overline{\mathbf{15}}, \mathbf{1})_+$

(20.156)

where the subscripts $\pm$ represent *intrinsic parities* (details can be found in [383]) ,

$$p = \gamma\phi\,. \tag{20.157}$$

p depends on the twist eigenvalue, γ, and the oscillator phase, ϕ. Note that $p = +$ for gauge and untwisted-sector states, and $p = +$ and $-$ have multiplicities 2 and 1 respectively for non-oscillator $T_{(01)}/T_{(02)}$ states.

Massless states in the untwisted and $T_{(0,1)}$, $T_{(0,2)}$ twisted sectors of the model are the intersections of those of the SO_{10} and $\mathrm{SU}_6 \times \mathrm{SU}_{2\mathrm{R}}$ models. This can be seen from the group branching rules. For example, the $T_{(0,1)}$-sector matter has the following branchings,

$$\begin{aligned} \mathrm{SO}_{10} &\to \mathrm{SU}_{4C} \otimes SU_{2L} \otimes SU_{2R} \\ \mathbf{16}_+ &= (\mathbf{4},\mathbf{2},\mathbf{1})_+ + (\overline{\mathbf{4}},\mathbf{1},\mathbf{2})_+ \,, \\ \mathbf{10}_- &= (\mathbf{6},\mathbf{1},\mathbf{1})_- + (\mathbf{1},\mathbf{2},\mathbf{2})_- \,, \end{aligned} \tag{20.158}$$

$$\begin{aligned} \mathrm{SU}_6 \times \mathrm{SU}_{2\mathrm{R}} &\to \mathrm{SU}_{4C} \otimes SU_{2L} \otimes SU_{2R} \\ (\overline{\mathbf{6}},\mathbf{2})_+ &= (\overline{\mathbf{4}},\mathbf{1},\mathbf{2})_+ + (\mathbf{1},\mathbf{2},\mathbf{2})_+ \,, \\ (\mathbf{15},\mathbf{1})_- &= (\mathbf{4},\mathbf{2},\mathbf{1})_- + (\mathbf{6},\mathbf{1},\mathbf{1})_- + (\mathbf{1},\mathbf{1},\mathbf{1})_- \,. \end{aligned} \tag{20.159}$$

The states in common,

$$2(\overline{\mathbf{4}},\mathbf{1},\mathbf{2})_+ + (\mathbf{6},\mathbf{1},\mathbf{1})_-, \tag{20.160}$$

agree with that of the T_2-twisted sector in Eq. (20.173).

Massless fields in the other, i.e. $T_{(1,2)}(= T_1)$ and $T_{(1,0)}(= T_3)$, twisted sectors are the unions of those of the SO_{10} and $\mathrm{SU}_6 \times \mathrm{SU}_{2\mathrm{R}}$ models (see Figs. 20.8 and 20.9). Therefore there are two sets of states, furnishing complete representations of SO_{10} and $\mathrm{SU}_6 \times \mathrm{SU}_{2\mathrm{R}}$ respectively. For example, the T_1 sector of the model contains

$$(\mathbf{4},\mathbf{2},\mathbf{1}) + (\overline{\mathbf{4}},\mathbf{1},\mathbf{2}) \quad \text{and} \quad (\mathbf{4},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1}), \tag{20.161}$$

which are in the complete representations $\mathbf{16}$ of SO_{10} and $(\mathbf{6},\mathbf{1})$ of $\mathrm{SU}_6 \times \mathrm{SU}_{2\mathrm{R}}$. These two sets of states have quantum numbers $n_2 = 0$ and $n_2 = 1$. (These quantum numbers are the winding numbers along the direction where the $\mathbf{W}_2$ Wilson line is

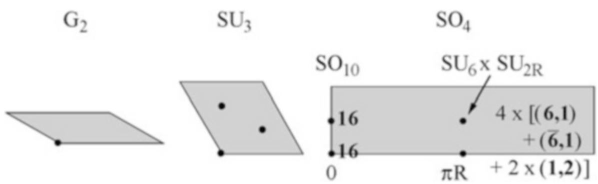

Fig. 20.8 $\mathrm{G}_2 \oplus \mathrm{SU}_3 \oplus \mathrm{SO}_4$ lattice with $\mathbb{Z}_6$ fixed points. The $T_{(1,1)}/T_{(1,2)}$ twisted sector states sit at these fixed points. *Reprinted from Nuclear Physics B 704, T. Kobayashi, S. Raby and R.J. Zhang, "Searching for realistic 4d string models with a Pati–Salam symmetry. Orbifold grand unified theories from heterotic string compactification on a $\mathbb{Z}_6$ orbifold," Page 16, Copyright (2005), with permission from Elsevier*

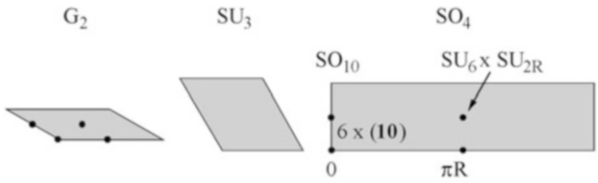

Fig. 20.9 $G_2 \oplus SU_3 \oplus SO_4$ lattice with $\mathbb{Z}_2$ fixed points. The $T_{(1,0)}$ twisted sector states sit at these fixed points. *Reprinted from Nuclear Physics B 704, T. Kobayashi, S. Raby and R.J. Zhang, "Searching for realistic 4d string models with a Pati–Salam symmetry. Orbifold grand unified theories from heterotic string compactification on a $\mathbb{Z}_6$ orbifold," Page 16, Copyright (2005), with permission from Elsevier*

imposed.) The $n_2 = 0$ and $n_2 = 1$ fixed points are thus the SO_{10} and $SU_6 \times SU_{2R}$ branes in the orbifold GUT language.

Identifying Orbifold Parities in String Theory

To a certain degree, the above E_6 heterotic model gives a string theoretical realization of the orbifold GUT in Chap. 16. Better yet, we also achieve an understanding of the orbifold parities in terms of string theoretical quantities. Specifically, the analogue of orbifold parities, Eq. (16.1), in our $\mathbb{Z}_6$ string models can be defined as follows

$$P = p e^{2\pi i(\mathbf{P}\cdot\mathbf{V}_2 - \mathbf{r}\cdot\mathbf{v}_2)}\,, \qquad P' = p e^{2\pi i(\mathbf{P}\cdot\mathbf{V}'_2 - \mathbf{r}\cdot\mathbf{v}_2)}\,, \tag{20.162}$$

where $\mathbf{V}_2$ and $\mathbf{V}'_2$ are the two inequivalent gauge embeddings of the $\mathbb{Z}_2$ twist in Sect. 20.4, and p is the intrinsic parity.

These parities can be deduced from the generalized Gliozzi-Scherk-Olive (GSO) projector [423, 431], as in the paragraphs after Eq. (20.150). Since the terms in the exponents, $\mathbf{P}\cdot\mathbf{V}_2 - \mathbf{r}\cdot\mathbf{v}_2$ and $\mathbf{P}\cdot\mathbf{V}'_2 - \mathbf{r}\cdot\mathbf{v}_2$, take integral or half-integral values, P and P' are either $+$ or $-$. The orbifold translation corresponds to the difference in P and P', i.e. $T = e^{2\pi i\mathbf{P}\cdot\mathbf{W}_2}$. The P, P' and T in string models have exactly the same properties as that of the orbifold GUTs.

Evidently, in the E_6 orbifold GUT model of Chap. 16, states supported at the SO_{10} and $SU_6 \times SU_{2R}$ branes are those with parities $P = +$ and $P' = +$, and states in the 4D effective theory are those with parities $P = P' = +$; this agrees with the string theoretical interpretation, since the parities in Eq. (20.162) are nothing but the required GSO projections for the gauge, untwisted and $T_{(01)}/T_{(02)}$ sector states (i.e. the bulk states) in string models. From information gathered in Sects. 20.4 and 20.5, we can also deduce the P and P' parities for the various bulk matter states. They are listed in Table 20.1. Note, KK masses for these bulk states can also be derived in string models as we will discuss in the next section.

Table 20.1 Parities for the bulk states in the model, computed from Eq. (20.162)

Multiplicities	States	P	P'	States	P	P'
1	$V(\mathbf{15},\mathbf{1},\mathbf{1})$	+	+	$\Sigma(\mathbf{15},\mathbf{1},\mathbf{1})$	−	−
1	$V(\mathbf{1},\mathbf{3},\mathbf{1})$	+	+	$\Sigma(\mathbf{1},\mathbf{3},\mathbf{1})$	−	−
1	$V(\mathbf{1},\mathbf{1},\mathbf{3})$	+	+	$\Sigma(\mathbf{1},\mathbf{1},\mathbf{3})$	−	−
1	$V(\mathbf{6},\mathbf{2},\mathbf{2})$	+	−	$\Sigma(\mathbf{6},\mathbf{2},\mathbf{2})$	−	+
1	$V(\mathbf{4},\mathbf{2},\mathbf{1})$	−	+	$\Sigma(\mathbf{4},\mathbf{2},\mathbf{1})$	+	−
1	$V(\overline{\mathbf{4}},\mathbf{1},\mathbf{2})$	−	−	$\Sigma(\overline{\mathbf{4}},\mathbf{1},\mathbf{2})$	+	+
1	$V(\overline{\mathbf{4}},\mathbf{2},\mathbf{1})$	−	+	$\Sigma(\overline{\mathbf{4}},\mathbf{2},\mathbf{1})$	+	−
1	$V(\mathbf{4},\mathbf{1},\mathbf{2})$	−	−	$\Sigma(\mathbf{4},\mathbf{1},\mathbf{2})$	+	+
1	$\mathbf{27}(\mathbf{4},\mathbf{2},\mathbf{1})$	+	+	$\overline{\mathbf{27}}(\overline{\mathbf{4}},\mathbf{2},\mathbf{1})$	−	−
1	$\mathbf{27}(\overline{\mathbf{4}},\mathbf{1},\mathbf{2})$	+	−	$\overline{\mathbf{27}}(\mathbf{4},\mathbf{1},\mathbf{2})$	−	+
1	$\mathbf{27}(\mathbf{6},\mathbf{1},\mathbf{1})$	−	+	$\overline{\mathbf{27}}(\mathbf{6},\mathbf{1},\mathbf{1})$	+	−
1	$\mathbf{27}(\mathbf{1},\mathbf{2},\mathbf{2})$	−	−	$\overline{\mathbf{27}}(\mathbf{1},\mathbf{2},\mathbf{2})$	+	+
2	$\mathbf{27}(\mathbf{4},\mathbf{2},\mathbf{1})_+$	+	−	$\overline{\mathbf{27}}(\overline{\mathbf{4}},\mathbf{2},\mathbf{1})_+$	−	+
2	$\mathbf{27}(\overline{\mathbf{4}},\mathbf{1},\mathbf{2})_+$	+	+	$\overline{\mathbf{27}}(\mathbf{4},\mathbf{1},\mathbf{2})_+$	−	−
2	$\mathbf{27}(\mathbf{6},\mathbf{1},\mathbf{1})_+$	−	−	$\overline{\mathbf{27}}(\mathbf{6},\mathbf{1},\mathbf{1})_+$	+	+
2	$\mathbf{27}(\mathbf{1},\mathbf{2},\mathbf{2})_+$	−	+	$\overline{\mathbf{27}}(\mathbf{1},\mathbf{2},\mathbf{2})_+$	+	−
1	$\mathbf{27}(\mathbf{4},\mathbf{2},\mathbf{1})_-$	−	+	$\overline{\mathbf{27}}(\overline{\mathbf{4}},\mathbf{2},\mathbf{1})_-$	+	−
1	$\mathbf{27}(\overline{\mathbf{4}},\mathbf{1},\mathbf{2})_-$	−	−	$\overline{\mathbf{27}}(\mathbf{4},\mathbf{1},\mathbf{2})_-$	+	+
1	$\mathbf{27}(\mathbf{6},\mathbf{1},\mathbf{1})_-$	+	+	$\overline{\mathbf{27}}(\mathbf{6},\mathbf{1},\mathbf{1})_-$	−	−
1	$\mathbf{27}(\mathbf{1},\mathbf{2},\mathbf{2})_-$	+	−	$\overline{\mathbf{27}}(\mathbf{1},\mathbf{2},\mathbf{2})_-$	−	+

The states have been decomposed to the PS irreducible representations. *Reprinted from Nuclear Physics B 704, T. Kobayashi, S. Raby and R.J. Zhang, "Searching for realistic 4d string models with a Pati–Salam symmetry. Orbifold grand unified theories from heterotic string compactification on a $\mathbb{Z}_6$ orbifold," Page 15, Copyright (2005), with permission from Elsevier*

As seen in Sect. 20.4, matter states in the $T_{(1,1)}/T_{(1,2)}$ and $T_{(1,0)}$ twisted sectors, which may be identified with the first two families, are localized on the two inequivalent fixed points in the SO_4 lattice. They are the SO_{10} and $SU_6 \times SU_{2R}$ brane states (see Figs. 20.8 and 20.9). These twisted-sector states are more tightly constrained than their orbifold GUT counterparts. In orbifold GUT models the only consistency requirement is the chiral anomaly cancellation, thus one can add arbitrary numbers of vector-like representations to the branes. String models have to satisfy more stringent modular invariance conditions (of course, one-loop modular invariance guarantees the model is anomaly free, up to a possible anomalous abelian factor [432–434]), which also constrains any additional matter in vector-like representations.

Summary: An Orbifold GUT—Heterotic String Dictionary

We first implement the $\mathbb{Z}_3$ sub-orbifold twist, which acts only on the G_2 and $SU(3)$ lattices. The resulting model is a 6D gauge theory with $N = 2$ hypermultiplet matter, from the untwisted and $T_{2,4}$ twisted sectors. This 6D theory is our starting point to reproduce the orbifold GUT models. The next step is to implement the $\mathbb{Z}_2$ sub-orbifold twist. The geometry of the extra dimensions closely resembles that of 6D orbifold GUTs. The $SO(4)$ lattice has four $\mathbb{Z}_2$ fixed points at 0, πR, $\pi R'$ and $\pi(R + R')$, where R and R' are on the e_5 and e_6 axes, respectively, of the lattice (see Figs. 20.4 and 20.6). When one varies the modulus parameter of the $SO(4)$ lattice such that the length of one axis (R) is much larger than the other (R') and the string length scale (ℓ_s), the lattice effectively becomes the $S^1/\mathbb{Z}_2$ orbi-circle in the 5D orbifold GUT, and the two fixed points at 0 and πR have degree-2 degeneracies. Furthermore, one may identify the states in the intermediate $\mathbb{Z}_3$ model, i.e. those of the untwisted and $T_{2,4}$ twisted sectors, as bulk states in the orbifold GUT.

Space-time supersymmetry and GUT breaking in string models work exactly as in the orbifold GUT models. First consider supersymmetry breaking. In the field theory, there are two gravitini in 4D, coming from the 5D (or 6D) gravitino. Only one linear combination is consistent with the space reversal, $y \to -y$; this breaks the $N = 2$ supersymmetry to that of $N = 1$. In string theory, the space-time supersymmetry currents are represented by those half-integral $SO(8)$ momenta.[21] The $\mathbb{Z}_3$ and $\mathbb{Z}_2$ projections remove all but two of them, $\mathbf{r} = \pm\frac{1}{2}(1, 1, 1, 1)$; this gives $N = 1$ supersymmetry in 4D.

Now consider GUT symmetry breaking. As usual, the $\mathbb{Z}_2$ orbifold twist and the translational symmetry of the $SO(4)$ lattice are realized in the gauge degrees of freedom by degree-2 gauge twists and Wilson lines respectively. To mimic the 5D orbifold GUT example, we impose only one degree-2 Wilson line, $\mathbf{W}_2$, along the long direction of the $SO(4)$ lattice, $\mathbf{R}$.[22] The gauge embeddings generally break the 5D/6D (bulk) gauge group further down to its subgroups, and the symmetry breaking works exactly as in the orbifold GUT models. This can clearly be seen from the following string theoretical realizations of the orbifold parities

$$P = p\, e^{2\pi i\,[\mathbf{P}\cdot\mathbf{V}_2 - \mathbf{r}\cdot\mathbf{v}_2]}\,, \quad P' = p\, e^{2\pi i\,[\mathbf{P}\cdot(\mathbf{V}_2+\mathbf{W}_2) - \mathbf{r}\cdot\mathbf{v}_2]}\,, \tag{20.163}$$

[21]Together with $\mathbf{r}_4 = (0, 0, 0, 1)$, they form the set of positive weights of the $\mathbf{8}_v$ representation of the $SO(8)$, the little group in 10d. $\pm\mathbf{r}_4$ represent the two uncompactified dimensions in the light-cone gauge. Their space-time fermionic partners have weights $\mathbf{r} = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ with even numbers of positive signs; they are in the $\mathbf{8}_s$ representation of $SO(8)$. In this notation, the fourth component of $\mathbf{v}_6$ is zero.

[22]Wilson lines can be used to reduce the number of chiral families. In all our models, we find it is sufficient to get three-generation models with two Wilson lines, one of degree 2 and one of degree 3. Note, however, that with two Wilson lines in the $SO(4)$ torus we can break $SO(10)$ directly to $SU(3) \times SU(2) \times U(1)_Y \times U(1)_X$ (see for example, [319, 435]).

where $\mathbf{V}_2 = 3\mathbf{V}_6$, and $p = \gamma\phi$ can be identified with intrinsic parities in the field theory language.[23] Since $2(\mathbf{P}\cdot\mathbf{V}_2 - \mathbf{r}\cdot\mathbf{v}_2)$, $2\mathbf{P}\cdot\mathbf{W}_2 = \mathbb{Z}$, by properties of the $E_8\times E_8$ and $SO(8)$ lattices, thus $P^2 = P'^2 = 1$, and Eq. (20.163) provides a representation of the orbifold parities. From the string theory point of view, $P = P' = +$ are nothing but the projection conditions, $\Delta = 1$, for the untwisted and $T_{2,4}$ twisted-sector states [see Eqs. (20.150)–(20.152)].

To reaffirm this identification, we compare the masses of KK excitations derived from string theory with that of orbifold GUTs. The coordinates of the $SO(4)$ lattice are untwisted under the $\mathbb{Z}_3$ action, so their mode expansions are the same as that of toroidal coordinates. Concentrating on the $\mathbf{R}$ direction, the bosonic coordinate is $X_{L,R} = x_{L,R} + p_{L,R}(\tau \pm \sigma) +$ oscillator terms, with p_L, p_R given by

$$p_L = \frac{m}{2R} + \left(1 - \frac{1}{4}|\mathbf{W}_2|^2\right)\frac{n_2 R}{\ell_s^2} + \frac{\mathbf{P}\cdot\mathbf{W}_2}{2R},$$
$$p_R = p_L - \frac{2n_2 R}{\ell_s^2}, \tag{20.164}$$

where m (n_2) are KK levels (winding numbers). The $\mathbb{Z}_2$ action maps m to $-m$, n_2 to $-n_2$ and $\mathbf{W}_2$ to $-\mathbf{W}_2$, so physical states must contain linear combinations, $|m, n_2\rangle \pm |-m, -n_2\rangle$; the eigenvalues ± 1 correspond to the first $\mathbb{Z}_2$ parity, P, of orbifold GUT models. The second orbifold parity, P', induces a non-trivial degree-2 Wilson line; it shifts the KK level by $m \to m + \mathbf{P}\cdot\mathbf{W}_2$. Since $2\mathbf{W}_2$ is a vector of the (integral) $E_8\times E_8$ lattice, the shift must be an integer or half-integer. When $R \gg R' \sim \ell_s$, the winding modes and the KK modes in the smaller dimension of $SO(4)$ decouple. Equation (20.164) then gives four types of KK excitations, reproducing the field theoretical mass formula in Eq. (14.57).

20.4.1 D_4 *Family Symmetry*

This general class of models has a D_4 family symmetry which constrains the possible Yukawa matrices for quarks and leptons. The third family is a bulk field, while the first two families are located on the two $\mathbb{Z}_2$ fixed points in the SO_4 torus with an SO_{10} gauge symmetry. One family sits at each fixed point (see Fig. 20.9). Since the Wilson line in the SO_4 torus lies in the orthogonal direction to these two fixed points, the theory is invariant under the permutation of the first two families, labeled by an index $n_2' = 0, 1$. In addition, the string selection rule requires that an effective superpotential given in terms of the fields associated with the twisted sector states at n_2' include an even number of fields at each fixed point with $n_2' = 0, 1$. Hence these effective superpotential terms are invariant under a

[23]For gauge and untwisted-sector states, p are trivial. For non-oscillator states in the $T_{2,4}$ twisted sectors, $p = \gamma$ are the eigenvalues of the G_2-plane fixed points under the $\mathbb{Z}_2$ twist. Note that $p = +$ and $-$ states have multiplicities 2 and 1 respectively since the corresponding numbers of fixed points in the G_2 plane are 2 and 1.

$\mathbb{Z}_2$ parity $f_{n'_2} \to -f_{n'_2}$. The two operations are generated by the two Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acting on a real two dimensional vector. The complete set of operations closes on the discrete non-abelian family symmetry group $\mathrm{D}_4 = \{\pm I,\ \pm\sigma_1,\ \pm\sigma_3,\ \mp i\sigma_2\}$.[24] Note that the eight-element finite (dihedral) group D_4 is the symmetry group of a square. It has five conjugacy classes and five faithful representations. The character table is

Classes	I	$-I$	$\pm\sigma_1$	$\pm\sigma_3$	$\mp i\sigma_2$
Doublet $-D$	2	-2	0	0	0
Singlet $-A_1$	1	1	1	1	1
Singlet $-B_1$	1	1	1	-1	-1
Singlet $-B_2$	1	1	-1	1	-1
Singlet $-A_2$	1	1	-1	-1	1

. (20.165)

In our models, the first two families transform as the doublet, while the third family and Higgs doublets transform as the trivial singlet.

We have many SO_{10} singlets in our models, transforming as doublets under D_4. They appear in effective higher dimension fermion mass operators [Eq. (20.168)]. Consider, for example, two doublets under D_4 given by $\{S_A,\ \tilde{S}_A\}$. Then in terms of these two doublets we can define bilinear combinations transforming as $\{A_1,\ A_2,\ B_1,\ B_2\}$. We have

$$\begin{aligned} S_1\,\tilde{S}_1 + S_2\,\tilde{S}_2 &\sim A_1 \\ S_1\,\tilde{S}_2 - S_2\,\tilde{S}_1 &\sim A_2 \\ S_1\,\tilde{S}_2 + S_2\,\tilde{S}_1 &\sim B_1 \\ S_1\,\tilde{S}_1 - S_2\,\tilde{S}_2 &\sim B_2 \end{aligned} \tag{20.166}$$

The effective Yukawa couplings are then constructed in terms of D_4 invariants. Define the D_4 doublet left-handed quarks and leptons (4,2,1) [$= \mathscr{Q}_A$] and left-handed anti-quarks and anti-leptons $(\bar{4}, 1, 2)$ [$= \bar{\mathscr{Q}}_A$] for the first two families and the Higgs multiplet (1,2,2) [$= \mathscr{H}$]. We then have the PS and D_4 invariants:

$$\begin{aligned} &\mathscr{H}\ \mathrm{A}_1\ (\mathscr{Q}_1\ \bar{\mathscr{Q}}_1 + \mathscr{Q}_2\ \bar{\mathscr{Q}}_2) \equiv \mathscr{H}\ \mathrm{A}_1\ (\mathscr{Q}_A\ \bar{\mathscr{Q}}_A) \\ &\mathscr{H}\ A_2\ (\mathscr{Q}_1\ \bar{\mathscr{Q}}_2 - \mathscr{Q}_2\ \bar{\mathscr{Q}}_1) \\ &\mathscr{H}\ B_1\ (\mathscr{Q}_1\ \bar{\mathscr{Q}}_2 + \mathscr{Q}_2\ \bar{\mathscr{Q}}_1) \\ &\mathscr{H}\ B_2\ (\mathscr{Q}_1\ \bar{\mathscr{Q}}_1 - \mathscr{Q}_2\ \bar{\mathscr{Q}}_2) \end{aligned} \tag{20.167}$$

[24] D_4 is a non-abelian subgroup of $SU(2)$. It is also equivalent to a subgroup of $O(2)$.

We can also have operators of the form

$$\mathscr{H}\,(\mathscr{Q}_A\,S_A)(\bar{\mathscr{Q}}_B\,S_B) = \mathscr{H}\,[\mathscr{Q}_1\,\bar{\mathscr{Q}}_1\,S_1^2 + \mathscr{Q}_2\,\bar{\mathscr{Q}}_2\,S_2^2 + (\mathscr{Q}_1\,\bar{\mathscr{Q}}_2 + \mathscr{Q}_2\,\bar{\mathscr{Q}}_1)\,S_1\,S_2] \tag{20.168}$$

Unfortunately there are, in principle, several possible ways of constructing D_4 invariants. We are not able to determine, without further string calculations, how to contract the D_4 indices. In [436] we studied flavor violation in the quark sector assuming this D_4 family symmetry. We found that the family symmetry can safely protect the theory against too large flavor violation.

Consistent with the D_4 family symmetry and the effective 5D orbifold GUT, we have a universal Yukawa coupling for the third family. It is given by the effective Lagrangian term

$$\mathscr{L}_3 = \frac{g_5}{\sqrt{\pi R}} \int_0^{\pi R} dy\,\overline{\mathbf{27}}\;\Sigma\;\mathbf{27} = g\bar{\mathscr{Q}}\,\mathscr{H}\,\mathscr{Q}. \tag{20.169}$$

20.5 Details of the Three-Family Pati-Salam Model

In this section we define the three-family PS model in the $\mathbb{Z}_6$ orbifold with $\mathbf{v}_6 = \frac{1}{6}(1, 2, -3)$. The $\mathbb{Z}_6$ is equivalent to the $\mathbb{Z}_2 \times \mathbb{Z}_3$ orbifold, where the two twist vectors are $\mathbf{v}_2 = 3\mathbf{v}_6 = \frac{1}{2}(1, 0, -1)$ and $\mathbf{v}_3 = 2\mathbf{v}_6 = \frac{1}{3}(1, -1, 0)$.

There are in total 61 inequivalent modular invariant choices for the gauge twists in the $\mathbb{Z}_6$ orbifold model [425, 426]. To narrow down the possibilities, we demand the models we start with (before imposing any Wilson line) contain an SO_{10} gauge group and some matter fields in $\mathbf{16}/\overline{\mathbf{16}}$ representations in the first or third twisted sectors. Although this step makes our results less generic, it greatly reduces the large number of possible models to a manageable subset. We choose the following gauge twist,

$$\mathbf{V}_6 = \frac{1}{6}\,(22200000)\,(11000000)\ , \tag{20.170}$$

which breaks the $E_8 \times E_8$ gauge symmetry down to $SO_{10} \times SU_3 \times E_7{}'$. The model contains four $\mathbf{16}$ and one $\overline{\mathbf{16}}$ in the untwisted sectors, and eighteen $\mathbf{16}$ and three $\overline{\mathbf{16}}$ in the twisted sectors; in total there are eighteen SO_{10} families [425, 426].

To further break the gauge symmetries and reduce the number of families, we impose discrete Wilson lines. As previously mentioned, there are at most one degree-3 Wilson line in the second complex plane, and two degree-2 Wilson lines in the third. We choose to add two of them, one of degree-2 and one of degree-3, as follows,

$$\begin{aligned}\mathbf{W}_2 &= \frac{1}{2}\,(10000111)\,(00000000) \\ \mathbf{W}_3 &= \frac{1}{3}\,(1-1000000)\,(00200000)\,.\end{aligned} \tag{20.171}$$

One can easily verify our choices satisfy the modular-invariance requirements, Eq. (20.145) and also the additional constraints given by

$$\{2(\mathbf{W}_2^{(i)})^2\,,\ 3(\mathbf{W}_3)^2\,,\ 4\mathbf{W}_2^{(1)}\cdot\mathbf{W}_2^{(2)}\,,\ 12\mathbf{W}_2^{(i)}\cdot\mathbf{W}_3\} = 0 \bmod 2. \tag{20.172}$$

The $\mathbf{W}_3$ Wilson line breaks the gauge group in the observable sector to SO_{10}, and the $\mathbf{W}_2$ breaks it further down to the PS gauge group.

The remaining unbroken gauge groups are $\mathrm{SU}_{4C}\otimes SU_{2L}\otimes SU_{2R}\times\mathrm{SO}'_{10}\times\mathrm{SU}'_2\times(\mathrm{U}_1)^5$ where the primed groups are in the hidden sector. The untwisted- and twisted-sector matter provide the following irreducible representations of the PS gauge group (modulo some singlets),

$$\begin{aligned}&U_1:\ (\mathbf{4},\mathbf{2},\mathbf{1})\,,\ U_2:(\mathbf{1},\mathbf{2},\mathbf{2})\,,\ U_3:(\mathbf{4},\mathbf{1},\mathbf{2})+(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})\,,\\ &T_1:\ 2(\mathbf{4},\mathbf{2},\mathbf{1})+2(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})+4(\mathbf{4},\mathbf{1},\mathbf{1})+4(\bar{\mathbf{4}},\mathbf{1},\mathbf{1})+8(\mathbf{1},\mathbf{2},\mathbf{1})+8(\mathbf{1},\mathbf{1},\mathbf{2})+2(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})\,,\\ &T_2:\ 2(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})+(\mathbf{6},\mathbf{1},\mathbf{1})\,,\qquad T_3:\ 6(\mathbf{6},\mathbf{1},\mathbf{1})+6(\mathbf{1},\mathbf{2},\mathbf{2})\,,\qquad T_4:\ (\mathbf{4},\mathbf{1},\mathbf{2})+2(\mathbf{6},\mathbf{1},\mathbf{1})\,,\end{aligned} \tag{20.173}$$

This model is an example of an E_6 orbifold GUT embedded into the heterotic string. In addition, it is an example of a model with a local $SO(10)$ GUT located in the T^1 twisted sector at the two fixed points located at the origin of the G_2 and SU(3) tori and on two fixed points on the SO(4) torus (see Fig. 20.8). Moreover at the two $SO(10)$ fixed points there are two complete families of quarks and leptons in the spinor representation of $SO(10)$.[25]

The complete massless matter spectrum for this model is given in Table 20.2. The model has many additional vector-like exotic states and additional $U(1)$ gauge groups, a large hidden sector non-abelian gauge group with hidden matter. Many of the SM singlet fields are moduli which can obtain VEVs without breaking the gauge symmetry. These VEVs can also give mass to the vector-like exotics, the extra $U(1)$ gauge bosons and generate non-trivial Yukawa couplings for quarks and leptons. A detailed study of the phenomenology of this model has not been performed. It is known that the model has the necessary states to spontaneously break PS to the SM. However, it is also known that in the process R-parity violating dimension 4 operators are generated. This is a major problem of the model. In the next chapter we discuss a class of MSSM models which have an exact R-parity.

[25] See problem 13.

Table 20.2 Massless matter spectrum

Sectors	$\mathrm{PS}\times\mathrm{SO}'_{10}\times\mathrm{SU}'_2$	Q_1	Q_2	Q_3	Q_4	Q_A	Labels	Sectors	$\mathrm{PS}\times\mathrm{SO}'_{10}\times\mathrm{SU}'_2$	Q_1	Q_2	Q_3	Q_4	Q_A	Labels
U_1	$(\mathbf{4},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	1	1	0	3	−2	f_3	$T_{2(-1)(1)(00)(000)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	−2	−4	−2	0	S_{14}
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{10},\mathbf{2})$	0	0	0	1	0	B'	$T_{2(-1)(2)(00)(000)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	4	−2	4	S_{15}
U_2	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$	2	−1	0	−2	2	h_1		$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	0	−2	−2	S_{16}
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	0	2	0	S_1		$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{10},\mathbf{1})$	0	0	−2	2	−2	A'_1
U_3	$(\mathbf{4},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	3	0	0	1	0	$\overline{\chi}^c_1$	$T_{2(1)(0)(00)(000)}$	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	1	0	0	1	0	$\chi^c_{1,2}$
	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	−3	0	0	−1	0	f^c_3	$T_{2(1)(1)(00)(000)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	−2	−2	−5	0	D'_5
$T_{1(1)(0)(0n')(000)}$	$(\mathbf{4},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	−1	0	0	0	0	$f_{1,2}$	$T_{2(1)(2)(00)(000)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	0	2	−1	4	D'_6
	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	−1	0	0	0	0	$f^c_{1,2}$		$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{16},\mathbf{1})$	0	0	−1	0	−2	F'
$T_{1(1)(1)(0n')(000)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	−2	0	−2	0	−2	D'_1	$T_{2(-1)(0)(00)(0\bar{1}0)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	2	4	4	2	S_{17}
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	2	1	2	2	2	D'_2	$T_{2(-1)(2)(00)(0\bar{1}0)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	0	0	−2	S_{18}
$T_{1(1)(2)(0n')(000)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	−2	1	2	0	0	D'_3	$T_{2(-1)(2)(00)(100)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	0	−2	1	−2	D'_7
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	2	−1	−2	−2	−2	D'_4	$T_{2(1)(0)(00)(100)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	2	4	4	2	S_{19}
$T_{1(1)(0)(1n')(000)}$	$(\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−1	0	2	−1	3	q_1	$T_{2(1)(2)(00)(100)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	0	0	−2	S_{20}
	$(\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−1	0	−2	1	−3	q_2	$T_{2(1)(2)(00)(0\bar{1}0)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	0	−2	1	−2	D'_8
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	2	0	2	0	3	D^ℓ_1	$T_{3(\omega)(0)(0n')(000)}$	$(\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	0	1	0	C_2
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	2	0	−2	2	−3	D^ℓ_2		$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$	0	0	0	1	0	h_2
$T_{1(1)(1)(1n')(000)}$	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	−2	0	2	−2	1	D^ℓ_3		$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	−1	0	−3	2	S_{21}
	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	0	−1	−2	−2	−3	D^r_1	$T_{3(\omega^2)(0)(0n')(000)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−2	1	0	1	−2	S_{22}
	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	1	0	2	−1	1	$\overline{q}_1$		$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	−1	0	−1	2	S_{23}
$T_{1(1)(2)(1n')(000)}$	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})$	0	0	0	−1	−1	Δ	$T_{3(1)(0)(0nv)(000)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−2	1	0	3	−2	S_{24}
$T_{1(1)(0)(0n')(100)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−2	−1	0	−3	2	S_2		$(\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	0	−1	0	C_3
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−2	−1	−4	−1	−4	S_3		$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$	0	0	0	−1	0	h_3
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−2	2	4	3	2	S_4	$T_{4(-1)(0)(00)(000)}$	$(\mathbf{4},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	−1	0	0	−1	0	$\overline{\chi}^c_2$

(continued)

Table 20.2 (continued)

Sectors	PS × SO$'_{10}$ × SU$'_2$	Q_1	Q_2	Q_3	Q_4	Q_A	Labels	Sectors	PS × SO$'_{10}$ × SU$'_2$	Q_1	Q_2	Q_3	Q_4	Q_A	Labels
$T_{1(1)(1)(0n')(100)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−2	0	0	−1	−2	S_5	$T_{4(-1)(1)(00)(000)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	2	2	5	0	D'_9
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	1	4	1	2	S_6	$T_{4(-1)(2)(00)(000)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	0	−2	1	−4	D'_{10}
$T_{1(1)(2)(0n')(100)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−2	1	0	3	0	S_7		$(\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{16}},\mathbf{1})$	0	0	1	0	2	$\overline{F}'$
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	−1	−4	1	−2	S_8	$T_{4(1)(0)(00)(000)}$	$(\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	0	0	0	0	C_4
$T_{1(1)(0)(1n')(100)}$	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	−2	−1	−2	−2	−1	D^ℓ_4		$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	−2	−4	−2	−2	S_{25}
	$(\overline{\mathbf{4}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	1	−1	−2	−1	−1	$\overline{q}_2$		$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−4	0	0	−2	0	S_{26}
$T_{1(1)(2)(1n')(100)}$	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	0	0	−2	2	−1	D^r_2	$T_{4(1)(1)(00)(000)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	2	4	2	0	S_{27}
$T_{1(1)(0)(1n')(010)}$	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	0	1	2	2	1	D^r_3	$T_{4(1)(2)(00)(000)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	−4	2	−4	S_{28}
$T_{1(1)(0)(0n')(001)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	0	0	1	0	S_9		$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	0	2	2	S_{29}
$T_{1(1)(0)(0n')(00\bar{1})}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	0	0	1	0	S'_9		$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{10},\mathbf{1})$	0	0	2	−2	2	A'_2
$T_{1(1)(0)(1n')(200)}$	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	0	1	2	2	1	D^r_4	$T_{4(-1)(0)(00)(\bar{1}00)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	−2	−4	−4	−2	S_{30}
$T_{1(1)(0)(0n')(110)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	0	0	1	0	S_{10}	$T_{4(-1)(2)(00)(\bar{1}00)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	0	0	2	S_{31}
$T_{1(1)(0)(0n')(300)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	0	0	1	0	S_{11}	$T_{4(-1)(2)(00)(010)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	0	2	−1	2	D'_{11}
$T_{2(-1)(0)(00)(000)}$	$(\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−2	0	0	0	0	C_1	$T_{4(1)(0)(00)(010)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	−2	−4	−4	−2	S_{32}
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	2	4	2	2	S_{12}	$T_{4(1)(2)(00)(010)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	0	0	2	S_{33}
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	4	0	0	2	0	S_{13}	$T_{4(1)(2)(00)(\bar{1}00)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	0	2	−1	2	D'_{12}

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Chapter 21
MSSM from the Heterotic String Compactified on $T^6/\mathbb{Z}_6$

This chapter is based on the work in [405–408, 437]. In this section we discuss just one "benchmark" model (Model 1) obtained via a "mini-landscape" search [405] of the $E_8 \times E_8$ heterotic string compactified on the $\mathbb{Z}_6$ orbifold [406].[1] The geometry of the model is the same as in the previous chapter. The difference is in the shift vector and Wilson lines. Moreover the goal is different. Here we look for an effective orbifold GUT which reduces to the Standard Model in 4D. The results of the search are quite remarkable. The benchmark model has many features consistent with a realistic MSSM. In particular, we show that all vector-like exotics and additional $U(1)$ gauge bosons can be lifted to high energy along F and D-flat directions. The model has an exact R parity which distinguishes Higgs doublets from lepton doublets and prevents dimension 4 baryon and lepton number violating operators. Right-handed neutrinos obtain large Majorana masses consistent with the standard See-Saw mechanism. Yukawa matrices, depending on the values of moduli VEVs, are hierarchical (but not physical) and there is a D_4 family symmetry. Finally the top quark Yukawa coupling is set by the GUT gauge coupling which is consistent with data.

[1]For earlier work on MSSM models from $\mathbb{Z}_6$ orbifolds of the heterotic string, see [413, 414]. Note, other MSSM-like Heterotic orbifold models have been obtained in the literature. For example, see [438].

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21.1 MSSM with R Parity

The model is defined by the shifts and Wilson lines

$$\begin{aligned} V &= \left(\frac{1}{3}, -\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0\right) \left(\frac{1}{2}, -\frac{1}{6}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) , \\ W_2 &= \left(0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right) \left(4, -3, -\frac{7}{2}, -4, -3, -\frac{7}{2}, -\frac{9}{2}, \frac{7}{2}\right) , \\ W_3 &= \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) \left(\frac{1}{3}, 0, 0, \frac{2}{3}, 0, \frac{5}{3}, -2, 0\right) . \end{aligned} \tag{21.1}$$

A possible second order 2 Wilson line is set to zero.

The shift V is defined to satisfy two criteria.

- The first criterion is the existence of a local $SO(10)$ GUT [2] at the T_1 fixed points at $x_6 = 0$ in the $SO(4)$ torus (Fig. 20.6).

$$P \cdot V = \mathbb{Z}; \;\; P \in SO(10) \;\; \text{momentum lattice.} \tag{21.2}$$

  Since the T_1 twisted sector has no invariant torus and only one Wilson line along the x_6 direction, all states located at these two fixed points must come in complete $SO(10)$ multiplets.
- The second criterion is that two massless spinor representations of $SO(10)$ are located at the $x_6 = 0$ fixed points.

Hence, the two complete families on the local $SO(10)$ GUT fixed points gives us an excellent starting point to find the MSSM. The Higgs doublets and third family of quarks and leptons must then come from elsewhere.

Let us now discuss the effective 5D orbifold GUT [439]. Consider the orbifold $(T^2)^3/\mathbb{Z}_3$ plus the Wilson line W_3 in the SU_3 torus. The $\mathbb{Z}_3$ twist does not act on the SO_4 torus, see Fig. 20.5. As a consequence of embedding the $\mathbb{Z}_3$ twist as a shift in the $E_8 \times E_8$ group lattice and taking into account the W_3 Wilson line, the first E_8 is broken to $SU(6)$. This gives the effective 5D orbifold gauge multiplet contained in the $N = 1$ vector field V. In addition we find the massless states

$$\Sigma \in \mathbf{35}, \quad \mathbf{20} + \mathbf{20}^c \quad \text{and} \quad 18(\mathbf{6} + \bar{\mathbf{6}}) \tag{21.3}$$

[2]For more discussion on local GUTs, see [412, 413].

in the 6D untwisted sector and T_2, T_4 twisted sectors. Together these form a complete $N = 2$ gauge multiplet

$$(V + \Sigma) \quad \text{and} \quad \mathbf{20} + 18(\mathbf{6}) \tag{21.4}$$

dimensional hypermultiplets. In fact the massless states in this sector can all be viewed as “bulk” states moving around in a large 5D space-time.

Now consider the $\mathbb{Z}_2$ twist and the Wilson line W_2 along the x_6 axis in the SO_4 torus. The action of the $\mathbb{Z}_2$ twist breaks the gauge group to $SU(5)$, while W_2 breaks $SU(5)$ further to the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$.

Let us now consider those MSSM states located in the bulk. From two of the pairs of $N = 1$ chiral multiplets $\mathbf{6} + \mathbf{6}^c$, which decompose as

$$\begin{aligned} &2 \times (\mathbf{6} + \mathbf{6}^c) \\ &\supset \left[(1, \mathbf{2})^{--}_{1,1} + (\mathbf{3}, \mathbf{1})^{-+}_{-2/3,1/3} \right] \\ &\quad + \left[(1, \mathbf{2})^{++}_{-1,-1} + (\bar{\mathbf{3}}, 1)^{--}_{2/3,-1/3} \right] \\ &\quad + \left[(1, \mathbf{2})^{-+}_{1,1} + (\mathbf{3}, \mathbf{1})^{--}_{-2/3,1/3} \right] \\ &\quad + \left[(1, \mathbf{2})^{+-}_{-1,-1} + (\bar{\mathbf{3}}, 1)^{++}_{2/3,-1/3} \right], \end{aligned} \tag{21.5}$$

we obtain the third family $\bar{b}$ and lepton doublet, l. The rest of the third family comes from the $\mathbf{10} + \mathbf{10}^c$ of $SU(5)$ contained in the $\mathbf{20} + \mathbf{20}^c$ of $SU(6)$, in the untwisted sector.

Now consider the Higgs bosons. The bulk gauge symmetry is $SU(6)$. Under $SU(5) \times U(1)$, the adjoint decomposes as

$$\mathbf{35} \to \mathbf{24}_0 + \mathbf{5}_{+1} + \mathbf{5}^c_{-1} + 1_0. \tag{21.6}$$

Thus the MSSM Higgs sector emerges from the breaking of the $SU(6)$ adjoint by the orbifold and the model satisfies the property of “gauge-Higgs unification.”

In the models with gauge-Higgs unification, the Higgs multiplets come from the 5D *vector* multiplet (V, Σ), both in the adjoint representation of $SU(6)$. V is the 4D gauge multiplet and the 4D chiral multiplet Σ contains the Higgs doublets. These states transform as follows under the orbifold parities

$(P\ P')$:

$$V : \left(\begin{array}{ccc|cc|c} (++) & (++) & (++) & (+-) & (+-) & (-+) \\ (++) & (++) & (++) & (+-) & (+-) & (-+) \\ (++) & (++) & (++) & (+-) & (+-) & (-+) \\ \hline (+-) & (+-) & (+-) & (++) & (++) & (--) \\ (+-) & (+-) & (+-) & (++) & (++) & (--) \\ \hline (-+) & (-+) & (-+) & (--) & (--) & (++) \end{array}\right) \tag{21.7}$$

$$\Sigma : \left(\begin{array}{ccc|cc|c} (--) & (--) & (--) & (-+) & (-+) & (+-) \\ (--) & (--) & (--) & (-+) & (-+) & (+-) \\ (--) & (--) & (--) & (-+) & (-+) & (+-) \\ \hline (-+) & (-+) & (-+) & (--) & (--) & (++) \\ (-+) & (-+) & (-+) & (--) & (--) & (++) \\ \hline (+-) & (+-) & (+-) & (++) & (++) & (--) \end{array}\right). \tag{21.8}$$

Hence, we have obtained doublet-triplet splitting via orbifolding.

21.2 D_4 Family Symmetry

The MSSM string model has a D_4 family symmetry with the first and second family transforming as a doublet and the third family and Higgs doublets transforming as D_4 singlets. We have:

- Since the top quarks and the Higgs are derived from the $SU(6)$ chiral adjoint and **20** hypermultiplet in the 5D bulk, they have a tree level Yukawa interaction given by

$$W \supset \frac{g_5}{\sqrt{\pi R}} \int_0^{\pi R} dy\, \mathbf{20^c}\ \Sigma\ \mathbf{20} = g_G\, Q_3\, H_u\, U_3^c \tag{21.9}$$

  where g_5 (g_G) is the 5D (4D) $SU(6)$ gauge coupling constant evaluated at the string scale. Further analysis on the top quark Yukawa coupling was done in [440] which analyzed corrections to the simple tree level result.
- The first two families reside at the $\mathbb{Z}_2$ fixed points, resulting in a D_4 family symmetry. Hence family symmetry breaking may be used to generate a hierarchy of fermion masses.[3]

[3]For a discussion of D_4 family symmetry and phenomenology, see [436]. For a general discussion of discrete non-Abelian family symmetries from orbifold compactifications of the heterotic string, see [384].

21.3 More Details of "Benchmark" Model 1 [406]

Let us now consider the spectrum, exotics, R parity, Yukawa couplings, and neutrino masses. In Table 21.1 we list the states of the model. In addition to the three families of quarks and leptons and one pair of Higgs doublets, we have vector-like exotics (states which can obtain mass without breaking any SM symmetry) and SM singlets. The SM singlets enter the superpotential in several important ways. They can give mass to the vector-like exotics via effective mass terms of the form

$$E E^c \tilde{S}^n \tag{21.10}$$

where E, E^c ($\tilde{S}$) represent the vector-like exotics and SM singlets respectively. We have checked that all vector-like exotics obtain mass at supersymmetric points in moduli space with $F = D = 0$. The SM singlets also generate effective Yukawa matrices for quarks and leptons, including neutrinos. In addition, the SM singlets give Majorana mass to the 16 right-handed neutrinos n_i^c, 13 conjugate neutrinos n_i and Dirac mass mixing the two. We have checked that the theory has only three light left-handed neutrinos.

However, one of the most important constraints in this construction is the existence of an exact low energy R parity. In this model we identified a generalized B-L (see Table 21.1) which is standard for the SM states and vector-like on the vector-like exotics. This $B-L$ naturally distinguishes the Higgs and lepton doublets.

Table 21.1 Spectrum. The quantum numbers under $SU(3) \times SU(2) \times [SU(4) \times SU(2)']$ are shown in boldface; hypercharge and B−L charge appear as subscripts

#	irrep	Label	#	irrep	Label
3	$(\mathbf{3},\mathbf{2};\mathbf{1},\mathbf{1})_{(1/3,1/3)}$	q_i	3	$(\overline{\mathbf{3}},\mathbf{1};\mathbf{1},\mathbf{1})_{(-4/3,-1/3)}$	$\bar{u}_i$
3	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_{(2,1)}$	$\bar{e}_i$	8	$(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})_{(0,*)}$	m_i
4	$(\overline{\mathbf{3}},\mathbf{1};\mathbf{1},\mathbf{1})_{(2/3,-1/3)}$	$\bar{d}_i$	1	$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})_{(-2/3,1/3)}$	$\bar{d}_i^c$
4	$(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})_{(-1,-1)}$	ℓ_i	1	$(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})_{(1,1)}$	ℓ_i^c
1	$(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})_{(-1,0)}$	ϕ_i	1	$(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})_{(1,0)}$	ϕ_i^c
6	$(\overline{\mathbf{3}},\mathbf{1};\mathbf{1},\mathbf{1})_{(2/3,2/3)}$	δ_i^c	6	$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})_{(-2/3,-2/3)}$	δ_i
14	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_{(1,*)}$	s_i^+	14	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_{(-1,*)}$	s_i^-
16	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_{(0,1)}$	$\bar{n}_i$	13	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_{(0,-1)}$	n_i
5	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})_{(0,1)}$	η_i^c	5	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})_{(0,-1)}$	η_i
10	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})_{(0,0)}$	h_i	2	$(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})_{(0,0)}$	y_i
6	$(\mathbf{1},\mathbf{1};\mathbf{4},\mathbf{1})_{(0,*)}$	f_i	6	$(\mathbf{1},\mathbf{1};\overline{\mathbf{4}},\mathbf{1})_{(0,*)}$	$\bar{\mathscr{Q}}_i$
2	$(\mathbf{1},\mathbf{1};\mathbf{4},\mathbf{1})_{(-1,-1)}$	f_i^-	2	$(\mathbf{1},\mathbf{1};\overline{\mathbf{4}},\mathbf{1})_{(1,1)}$	$\bar{\mathscr{Q}}_i^+$
4	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_{(0,\pm 2)}$	χ_i	32	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_{(0,0)}$	s_i^0
2	$(\overline{\mathbf{3}},\mathbf{1};\mathbf{1},\mathbf{1})_{(-1/3,2/3)}$	ξ_i^c	2	$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})_{(1/3,-2/3)}$	ξ_i

Note that the states $s_i^{\pm}, f_i, \bar{f}_i$ and m_i have different $B-L$ charges for different i, which we do not explicitly list [406]

Moreover we found SM singlet states

$$\tilde{S} = \{h_i,\ \chi_i,\ s_i^0\} \tag{21.11}$$

which can get vacuum expectation values preserving a matter parity $\mathbb{Z}_2^{\mathcal{M}}$ subgroup of $U(1)_{B-L}$. It is this set of SM singlets which give vector-like exotics mass and effective Yukawa matrices for quarks and leptons. In addition, the states χ_i give Majorana mass to neutrinos.

As a final note, we have evaluated the μ term in this model. As a consequence of gauge-Higgs unification, the product $H_u H_d$ is a singlet under all $U(1)$s. Moreover, it is also invariant under all string selection rules, i.e. H-momentum and space-group selection constraints. As a result the μ term is of the form

$$\mu\ H_u\ H_d = W_0(\tilde{S})\ H_u\ H_d \tag{21.12}$$

where the factor $W_0(\tilde{S})(=\mu)$ is a polynomial in SM singlets and includes all terms which can also appear in the superpotential for the SM singlet fields, $\widetilde{W}_0(\tilde{S})$. Thus when we demand a flat space supersymmetric limit, we are also forced to $\mu \equiv W_0(\tilde{S}) = \widetilde{W}_0\tilde{S} = 0$[4], i.e. μ vanishes in the flat space supersymmetric limit. This is encouraging, since when SUSY is broken we expect both terms to be non-vanishing and of order the weak scale.

21.4 Summary

At this point it might be worthwhile to summarize the progress in string constructions of theories which look like the real world. So far we have models which contain the correct gauge groups and matter representations. This is a great success. This success requires embedding orbifold GUTs into the heterotic string with matter in complete GUT representations sitting a local GUT fixed points. The models include a family symmetry which constrains the fermion mass hierarchy as well as flavor violating interactions. They also allow for Majorana masses for right-handed neutrinos, and thus the standard See-Saw mechanism. In addition, the models have an exact R-parity which forbids dimension 4 baryon and lepton number violating operators and the lightest SUSY particle is a stable candidate for dark matter.

So what are the remaining problems (and there are many!). So far the models allow for dimension 5 baryon and lepton number violating operators which are not

[4]We have not shown that the coefficients of the individual monomials in $W_0(\tilde{S})$ are, in general, identical in both the μ term and in the SM singlet superpotential term, $\widetilde{W}_0\tilde{S}$. Nevertheless at sixth order in SM singlet fields we have shown that when one vanishes, so does the other. This is because each monomial contains a bi-linear in D_4 doublets and this family symmetry fixes the relative coefficient in the product. Therefore when the product of D_4 doublets vanishes, we have $\mu = W_0(\tilde{S}) = \widetilde{W}_0\tilde{S}. = 0$.

sufficiently suppressed. Thus rapid proton decay is a problem. We address a solution to this problem in Chap. 23. In addition, GUT symmetry breaking via Wilson lines is local. It would be nice to have examples where GUT symmetry breaking was global, such that the compactification scale is the same as the GUT scale. This concern is also addressed in Chap. 23.

However, the most serious problems are related to dealing with all the moduli and breaking supersymmetry. At the present time, all gauge and Yukawa couplings are functions of moduli VEVs. The moduli lie along flat directions in the potential. These flat directions are typically only lifted once SUSY is broken. Thus the real fine tuning problem is how is it possible for all the moduli to obtain the precise values necessary to generate gauge and Yukawa couplings consistent with low energy data. And at the same time the cosmological constant needs to be close to zero to one part in 10^{120} in Planck units. Moreover, we have assumed from the start that six space dimensions are compactified. So the question becomes, what is special about six compact dimensions. A possible answer is given in [441].

Finally, the orbifold models we have discussed contain orbifold fixed points, which are singular geometries with infinite space-time curvature localized at the fixed points. There is no problem with these fixed points for the two dimensional conformal field theory which defines the string. However, in order to obtain a well-defined supergravity limit, one would like to smooth out the singular points. This requires, so-called “blowup modes” (massless chiral singlets corresponding to massless twisted sector states) at the fixed points. In the model presented here, there are some fixed points which only contain states which are non-singlets under the Standard Model gauge group. Therefore these fixed points cannot be blown up without breaking at least hypercharge [442, 443]. This is not a problem for the string theory per se, but it prevents the realization of a smooth supergravity limit.

In the next chapter, we discuss the issues of gauge coupling unification and proton decay in the heterotic string models considered here.

Chapter 22
Gauge Coupling Unification and Proton Decay

This chapter is based on the work of [439]. We have checked whether the SM gauge couplings unify at the string scale in the class of models similar to Model 1 discussed in the previous chapter. All of the 15 MSSM-like models of [406] have three families of quarks and leptons and one or more pairs of Higgs doublets. They all admit an $SU(6)$ orbifold GUT with gauge-Higgs unification and the third family in the bulk. They differ, however, in other bulk and brane exotic states. We show that the KK modes of the model, including only those of the third family and the gauge sector, are *not* consistent with gauge coupling unification at the string scale. Nevertheless, we show that it is possible to obtain unification if one adjusts the spectrum of vector-like exotics below the compactification scale. As an example, see Fig. 22.1. Note, the compactification scale is less than the 4D GUT scale and some exotics have mass two orders of magnitude less than M_C, while all others are taken to have mass at $M_{\rm STRING}$. In addition, the value of the GUT coupling at the string scale, $\alpha_G(M_{STRING}) \equiv \alpha_{string}$, satisfies the weakly coupled heterotic string relation

$$G_N = \frac{1}{8}\,\alpha_{string}\,\alpha' \tag{22.1}$$

or

$$\alpha_{string}^{-1} = \frac{1}{8}\,(\frac{M_{Pl}}{M_{\rm STRING}})^2. \tag{22.2}$$

In Fig. 22.2 we plot the distribution of solutions with different choices of light exotics. On the same plot we give the proton lifetime due to dimension 6 operators. Recall in these models the two light families are located on the $SU(5)$ branes, thus the proton decay rate is only suppressed by the compactification scale, M_C^{-2}. Note, 90% of the models are already excluded by the Super-Kamiokande bounds on the proton lifetime. The remaining models may be tested at a next generation megaton water čerenkov detector or at the Liquid Argon detector proposed for DUNE.

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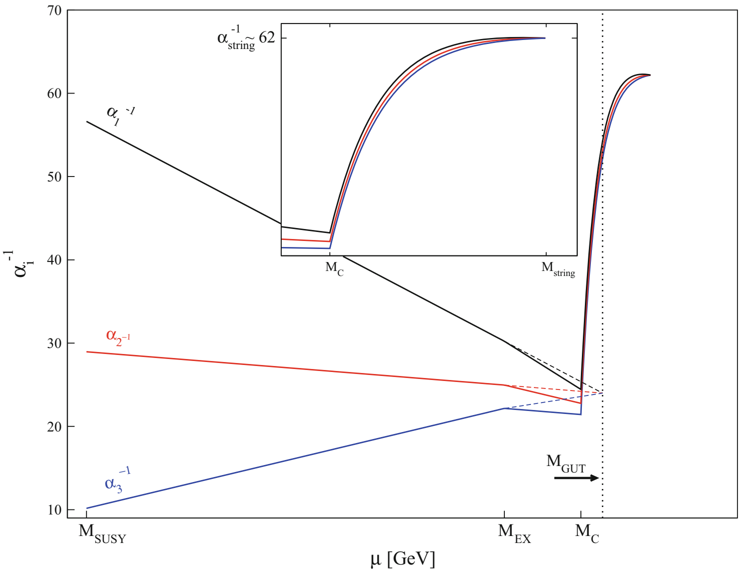

Fig. 22.1 An example of the type of gauge coupling evolution we see in these models, versus the typical behavior in the MSSM. The "tail" is due to the power law running of the couplings when towers of Kaluza-Klein modes are involved. Unification in this model occurs at $M_{\text{STRING}} \simeq 5.5 \times 10^{17}$ GeV, with a compactification scale of $M_C \simeq 8.2 \times 10^{15}$ GeV, and an exotic mass scale of $M_{\text{EX}} \simeq 8.2 \times 10^{13}$ GeV

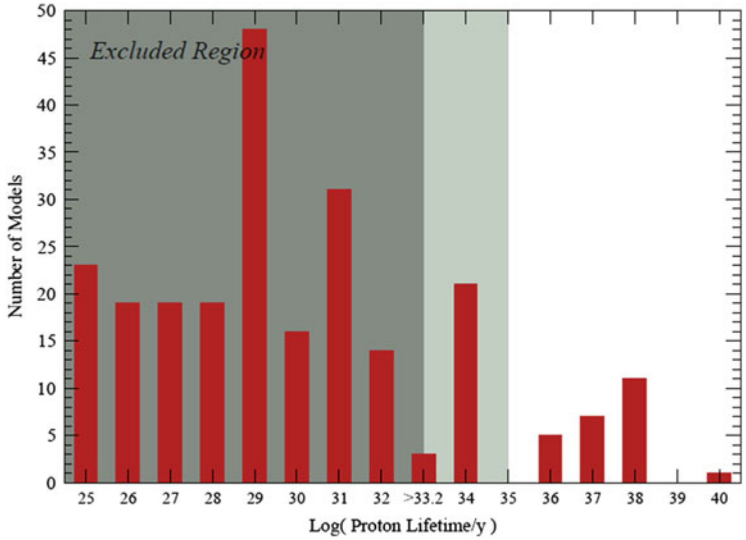

Fig. 22.2 Histogram of solutions with $M_{\text{STRING}} > M_C \gtrsim M_{\text{EX}}$, showing the models which are excluded by Super-K bounds (*darker grey*) and those which are potentially accessible in a next generation proton decay experiment (*lighter grey*). Of 252 total solutions, 48 are not experimentally ruled out by the current experimental bound, and most of the remaining parameter space can be eliminated in the next generation of proposed proton decay searches

Chapter 23
String Theory Realization of $\mathbb{Z}_4^R$ Symmetry

In Chap. 19 we showed that discrete R symmetries can be used to define the MSSM. In this chapter we present a globally consistent string compactification with the exact MSSM spectrum below the compactification scale. The model exhibits the $\mathbb{Z}_4^R$ symmetry, which originates from the Lorentz group of compactified dimensions [276]. In the discussion of Sect. 19.2 we argued that, if some hidden sector strong dynamics was responsible for supersymmetry breakdown, also a μ term of the right size will be induced by this dynamics. In order to render our discussion more specific, we will now discuss an explicit, globally consistent string-derived model. Such models have the important property that they are complete, i.e. unlike bottom-up (or 'local') models they cannot be 'amended' by some extra states or sectors. This allows us to clarify whether or not a reasonable μ term will appear.

Making extensive use of the methods to determine the remnant symmetries described in [444], we were able to find examples realizing the $\mathbb{Z}_4^R$ discussed in Sect. 19.2, based on the string model derived in [445] and similar models, with the exact MSSM spectrum, a large top Yukawa coupling, a non-trivial hidden sector etc. In what follows, we present an explicit example.

General Picture

String theory compactifications provide us with a plethora of vacuum configurations, each of which comes with symmetries and, as a consequence, with extra massless degrees of freedom whose mass terms are prohibited by these symmetries. Simple examples for such compactifications include heterotic orbifolds, [398, 399], where the rank of the gauge group after compactification equals that of $E_8 \times E_8$, i.e. 16. A few hundreds of orbifold models are known in which $E_8 \times E_8$ gets broken to the standard model gauge symmetry $G_{\mathrm{SM}} = \mathrm{SU}(3)_C \times \mathrm{SU}(2)_{\mathrm{L}} \times U(1)_Y$ (with hypercharge in GUT normalization) times $U(1)^n$ times a hidden sector group and the chiral spectra of the MSSM [405, 407]. They also exhibit exotics which are

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vector-like with respect to G_{SM} and which can be decoupled when the extra gauge symmetries are broken. Each of these models contains many vacua, i.e. solutions of the supersymmetry conditions $V_F = V_D = 0$. Typically these vacua exhibit flat directions before supersymmetry breaking.

At the orbifold point, where the vacuum expectation values (VEVs) of all fields are zero, we have discrete R as well as continuous and discrete non-R symmetries. Typically one of the U(1) symmetries appears anomalous, which is conventionally denoted by $U(1)_{\text{anom}}$. Also some of the discrete symmetries may appear anomalous [390, 446]. After assigning VEVs to certain fields, some of the symmetries are spontaneously broken and others remain. We shall be mainly interested in remnant discrete symmetries, which can be of R or non-R type and be either anomalous or non-anomalous. We will discuss examples of all kinds in Sect. 23.1.

Clearly, one cannot assign VEVs to the fields at will. Rather, one has to identify field configurations which correspond to local minima of the (effective) scalar potential. Let us briefly describe the first steps towards identifying such vacuum configurations. Consider a configuration in which several fields attain VEVs. We focus on "maximal vacua" (as in [447]), i.e. we assume that all fields which are neutral under the remnant gauge and discrete symmetries, called $\phi^{(i)}$ $(1 \leq i \leq N)$ in what follows, attain VEVs (if these are consistent with D-flatness). All fields without expectation value, denoted by $\psi^{(j)}$ $(1 \leq j \leq M)$, therefore transform non-trivially under some of the remnant symmetries.

Discrete Non-R Symmetries

The case of vacua with non-R discrete symmetries has been discussed in detail in [414, 447]. In this case, the superpotential has the form

$$\mathscr{W} \;=\; \Omega(\phi^{(1)}, \dots \phi^{(N)}) + (\text{terms at least quadratic in the } \psi^{(j)}) \;. \tag{23.1}$$

Therefore, the F-term equations for the $\psi^{(j)}$ fields trivially vanish and we are left with N F-term equations for the N $\phi^{(i)}$ fields, which generically have solutions. Hence, if all $\phi^{(i)}$ enter gauge invariant monomials composed of $\phi^{(i)}$ fields only, we will find supersymmetric vacua, i.e. solutions to the F- and D-term equations.

Because of the above arguments it is sufficient to look at the system of $\phi^{(i)}$ fields only, which has been studied in the literature. Consider the case of a *generic* superpotential $\mathscr{W}$. It is known that the solutions to the D- and F-term equations intersect *generically* in a point [448]. That is, there are point-like field configurations which satisfy

$$D_a \;=\; F_i \;=\; 0 \quad \text{at}\, \phi^{(i)} \;=\; \langle \phi^{(i)} \rangle \;, \tag{23.2}$$

where, as usual,

$$D_a = \sum_i (\phi^{(i)})^* \mathsf{T}_a \phi^{(i)} , \tag{23.3a}$$

$$F^{(i)} = \frac{\partial \mathscr{W}}{\partial \phi^{(i)}} . \tag{23.3b}$$

The term 'point-like' means that there are no massless deformations of the vacuum (23.2). The reason why these vacua are point-like is easily understood: generically the F-term equations constitute as many gauge invariant constraints as there are gauge invariant variables. However, this also means that, at least generically,

$$\mathscr{W}|_{\phi^{(i)} = \langle\phi^{(i)}\rangle} \neq 0 . \tag{23.4}$$

If the fields attain VEVs $\langle\phi^{(i)}\rangle$ of the order of the fundamental scale, one hence expects to have too large a VEV for $\mathscr{W}$. One possible solution to the problem relies on approximate R symmetries [371], where one obtains a highly suppressed VEV of the superpotential. In what follows, we discuss an alternative: in settings with a residual R symmetry the above conclusion can be avoided as well.

Discrete R Symmetries

Let us now discuss vacua with discrete R symmetries. To be specific, consider the order four symmetry $\mathbb{Z}_4^R$, under which the superpotential $\mathscr{W}$ has charge 2, such that

$$\mathscr{W} \xrightarrow{\zeta} -\mathscr{W} \tag{23.5}$$

under the $\mathbb{Z}_4^R$ generator ζ. Superspace coordinates transform as

$$\theta_\alpha \rightarrow \mathrm{i}\,\theta_\alpha \tag{23.6}$$

such that the F-term Lagrangian

$$\mathscr{L}_F = \int \mathrm{d}^2\theta\, \mathscr{W} + \text{h.c.} \tag{23.7}$$

is invariant. Chiral superfields will have R charges $0, 1, 2, 3$.[1] Both the fields of the type ψ_1 and ψ_3 with R charges 1 and 3, respectively, can acquire mass as the ψ_1^2 and ψ_3^2 terms have R charge $2 \mod 4$ and thus denote allowed superpotential terms.

[1] A special role is played by the dilaton S, whose imaginary part $a = \operatorname{Im} S|_{\theta=0}$ shifts under $\mathbb{Z}_4^R$.

The system of fields $\phi_0^{(i)}$ and $\psi_2^{(j)}$ with R charges 0 and 2, respectively, is more interesting. Consider first only one field ϕ_0 and one field ψ_2. The structure of the superpotential is

$$\mathscr{W} = \psi_2 \cdot f(\phi_0) + \mathscr{O}(\psi_2^3) \tag{23.8}$$

with some function f. The F-term for ϕ_0 vanishes trivially as long as $\mathbb{Z}_4^R$ is unbroken,

$$\frac{\partial \mathscr{W}}{\partial \phi_0} = \psi_2 \cdot f'(\phi_0) = 0\,. \tag{23.9}$$

Note that due to the $\mathbb{Z}_4^R$ symmetry the superpotential vanishes in the vacuum. Thus it is sufficient to look at the global supersymmetry F-terms. On the other hand, the F-term constraint (at $\psi_2 = 0$)

$$\frac{\partial \mathscr{W}}{\partial \psi_2} = f(\phi_0) \stackrel{!}{=} 0 \tag{23.10}$$

will in general fix ϕ_0 at some non-trivial zero $\langle \phi_0 \rangle$ of f. Indeed, there will be a supersymmetric mass term, which can be seen by expanding ϕ_0 around its VEV, i.e. inserting $\phi_0 = \langle \phi_0 \rangle + \delta\phi_0$ into (23.8),

$$\mathscr{W} = f'(\langle \phi_0 \rangle)\, \delta\phi_0\, \psi_2 + \mathscr{O}(\delta\phi_0^2, \psi_2^3)\,. \tag{23.11}$$

The supersymmetric mass $f'(\langle \phi_0 \rangle)$ is generically different from 0.

Repeating this analysis for N $\phi_0^{(i)}$ and M $\psi_2^{(j)}$ fields reveals that the F-terms of the $\psi_2^{(j)}$ lead to M, in general independent, constraints on the $\phi_0^{(i)}$ VEVs. For $N = M$ we therefore expect point-like vacua with all directions fixed in a supersymmetric way.

To summarize, systems with a residual R symmetry ensure, unlike in the case without residual symmetries, that $\langle \mathscr{W} \rangle = 0$. However, in systems which exhibit a linearly realized $\mathbb{Z}_4^R$ somewhere in field space it may not be possible to find a supersymmetric vacuum that preserves $\mathbb{Z}_4^R$. In the case of a generic superpotential this happens if there are more, i.e. $M > N$, fields with R charge 2 than with 0. On the other hand, if there are more fields with R charge 0 than with 2, i.e. for $M < N$, one expects to have a Minkowski vacuum with $N - M$ flat directions. For $N = M$ one can have supersymmetric Minkowski vacua with all directions fixed in a supersymmetric way.

An important comment in this context concerns the moduli-dependence of couplings. As we have seen, in the case of discrete R-symmetries one might obtain more constraint (i.e. F-term) equations than R-even 'matter' fields. Specifically, in string vacua one should, however, carefully take into account all R-even fields, also the Kähler and complex structure moduli, T_i and U_j, on whose values the coupling strengths depend.

23.1 An Explicit String-Derived Model

In order to render our discussion more specific, we base our analysis on a concrete model (see [276]). We consider a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold compactification with an additional freely acting $\mathbb{Z}_2$ of the $E_8 \times E_8$ heterotic string. Details of the model including shift vectors and Wilson lines can be found in Sect. 23.3.

In [445] a vacuum configuration of a very similar $\mathbb{Z}_2 \times \mathbb{Z}_2$ model with matter parity and other desirable features was presented. However, the vacuum configuration discussed there has the unpleasant property that, at least generically, all Higgs fields attain large masses. In what follows we discuss how this can be avoided by identifying vacuum configurations with enhanced symmetries. In [353] another vacuum with the $\mathbb{Z}_4^R$ symmetry discussed in the introduction was found by using the methods presented in this paper. In both models the GUT symmetry is broken non-locally. This may be advantageous from the point of view of precision gauge unification [343]. It also avoids fractionally charged exotics, which appear in many other compactifications (cf. the discussion in [449]).

Labeling of States

We start our discussion with a comment on our notation. In a first step, we label the fields according to their $G_{\mathrm{SM}} \times [\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(2)]_{\mathrm{hid}}$ quantum numbers. In particular, we denote the standard model representations with lepton/Higgs and d-quark quantum numbers as

$$L_i : (\mathbf{1}, \mathbf{2})_{-1} , \tag{23.12a}$$

$$\bar{L}_i : (\mathbf{1}, \mathbf{2})_{1} , \tag{23.12b}$$

$$D_i : (\mathbf{3}, \mathbf{1})_{-2/3} , \tag{23.12c}$$

$$\bar{D}_i : (\overline{\mathbf{3}}, \mathbf{1})_{2/3} . \tag{23.12d}$$

In the next step we identify $\mathbb{Z}_4^R$ such that the $\bar{L}_i/L_i$ decompose in lepton doublets ℓ_i with odd $\mathbb{Z}_4^R$ charges and Higgs candidates h_d/h_u with even $\mathbb{Z}_4^R$ charges etc. The details of labeling states are given in Sect. 23.3.

VEV Configuration

Following the steps discussed in [444], we obtained a promising configuration in which the fields

$$\begin{aligned}\widetilde{\phi}^{(i)} = \{&\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8, \phi_9, \phi_{10}, \phi_{11}, \phi_{12}, \phi_{13}, \phi_{14},\\ &x_1, x_2, x_3, x_4, x_5, \bar{x}_1, \bar{x}_3, \bar{x}_4, \bar{x}_5, y_3, y_4, y_5, y_6\}\end{aligned} \tag{23.13}$$

attain VEVs. The full quantum numbers of these fields are given in Table 23.3 in Sect. 23.3. In order to ensure D-flatness with respect to the hidden sector gauge factors, in a given basis not all components of the $x_i/\bar{x}_i$ and y_i attain VEVs. Details are given in Appendix D [276].

Remnant Discrete Symmetries

By giving VEVs to the $\widetilde{\phi}^{(i)}$ fields in (23.13), we arrive at a vacuum in which, apart from G_{SM} and a 'hidden' SU(2), all gauge factors are spontaneously broken. The vacuum exhibits a $\mathbb{Z}_4^R$ symmetry, whereby the superpotential $\mathscr{W}$ has $\mathbb{Z}_4^R$ charge 2.

The $\mathbb{Z}_4^R$ charges of the matter fields are shown in Table 23.1. The detailed origin of the $\mathbb{Z}_4^R$ symmetry is discussed later. Given these charges, we confirm by a straightforward field-theoretic calculation (cf. [359, 390]) that $\mathbb{Z}_4^R$ appears indeed anomalous with universal $\mathrm{SU}(2)_{\mathrm{L}} - \mathrm{SU}(2)_{\mathrm{L}} - \mathbb{Z}_4^R$ and $\mathrm{SU}(3)_C - \mathrm{SU}(3)_C - \mathbb{Z}_4^R$ anomalies (see [353] and Appendix A [276]). The statement that $\mathbb{Z}_4^R$ appears anomalous means, as we shall discuss in detail below, that the anomalies are cancelled by a Green-Schwarz (GS) mechanism. On the other hand, the $\mathbb{Z}_4^R$ has a, by the traditional criteria, non-anomalous $\mathbb{Z}_2^{\mathscr{M}}$ subgroup which is equivalent to matter parity [353].

D-Flatness

As already discussed, we cannot switch on the $\widetilde{\phi}^{(i)}$ fields at will; rather we have to show that there are vacuum configurations in which all these fields acquire VEVs. This requires to verify that the D- and F-term potentials vanish. With the Hilbert basis method (see Appendix C [276]) a complete set of D-flat directions is identified composed of $\widetilde{\phi}^{(i)}$ fields. The dimension of the D-flat moduli space is evaluated using Singular [450] and the STRINGVACUA [451] package; the result is that there

Table 23.1 $\mathbb{Z}_4^R$ charges of the (a) matter fields and (b) Higgs and exotics

(a) Quarks and leptons

	q_i	$\bar{u}_i$	$\bar{d}_i$	ℓ_i	$\bar{e}_i$
$\mathbb{Z}_4^R$	1	1	1	1	1

(b) Higgs and exotics

	h_1	h_2	h_3	h_4	h_5	h_6	$\bar{h}_1$	$\bar{h}_2$	$\bar{h}_3$	$\bar{h}_4$	$\bar{h}_5$	$\bar{h}_6$	δ_1	δ_2	δ_3	$\bar{\delta}_1$	$\bar{\delta}_2$	$\bar{\delta}_3$
$\mathbb{Z}_4^R$	0	2	0	2	0	0	0	2	0	0	2	2	0	2	2	2	0	0

The index i in (a) takes values $i = 1, 2, 3$. *Reprinted from Nuclear Physics B 847, R. Kappl, B. Petersen, S. Raby, M. Ratz, R. Schieren, and P.K.S. Vaudrevange, "String-derived MSSM vacua with residual R symmetries," Page 331, Copyright (2011), with permission from Elsevier* are 18 D-flat directions; the details of the computation are collected in Appendix D [276].

F-Term Constraints

Next we consider the F-term constraints. As discussed in Sect. 23, the F-term conditions come from the fields with R-charge 2. The number of independent conditions is computed in Appendix D [276]. The result is that there are 23 independent conditions on $18 + 6 = 24$ D-flat directions, where the Kähler and complex structure moduli are included. We therefore expect to find supersymmetric vacuum configurations in which all the $\widetilde{\phi}^{(i)}$ acquire VEVs. In this configuration, almost all singlet fields, including the geometric moduli are fixed in a supersymmetric way. It will be interesting to compare this result to similar results found recently in the context of smooth heterotic compactifications [452]. We expect a significantly different, i.e. healthier, phenomenology than in the case in which a large number of singlets acquire mass only after supersymmetry breaking [453, 454]. Notice that there are two possible caveats. First, the analysis performed strictly applies only to superpotentials which are, apart from all the symmetries we discuss, generic. Second, it might happen that there are supersymmetric vacua, but they occur at large VEVs of some of the fields, i.e. in regions of field space where we no longer control our construction. Both issues will be addressed elsewhere.

Higgs vs. Matter

The $\mathbb{Z}_2^{\mathscr{M}}$ subgroup of the $\mathbb{Z}_4^R$ symmetry allows us to discriminate between

- 3 lepton doublets, $\ell_i = \{L_4, L_6, L_7\}$,
- 3 d-type quarks, $\bar{d}_i = \{\bar{D}_1, \bar{D}_3, \bar{D}_4\}$,

on the one hand, and

- Higgs candidates, $h_i = \{L_1, L_2, L_3, L_5, L_8, L_9\}$ and $\bar{h}_i = \{\bar{L}_1, \bar{L}_2, \bar{L}_3, \bar{L}_4, \bar{L}_5, \bar{L}_6\}$,
- exotic triplets, $\delta_i = \{D_1, D_2, D_3\}$ and $\bar{\delta}_i = \{\bar{D}_2, \bar{D}_5, \bar{D}_6\}$

on the other hand.

Decoupling of Exotics

With the charges in Table 23.1 we can readily analyze the structure of the mass matrices. We crosscheck these structures by explicitly computing the couplings allowed by the string selection rules (cf. [445]). Note there is a caveat: our results are based on the assumption that all couplings that are allowed by the selection rules will appear with a non-vanishing coefficient. A $\widetilde{\phi}^n$ in the matrices represents a known polynomial of order n in the $\widetilde{\phi}$ fields which is calculated using string selection rules. A zero entry in the matrices means that the corresponding coupling is not present in the perturbative superpotential. The $\bar{h}_i - h_j$ Higgs mass matrix is

$$\mathscr{M}_h = \begin{pmatrix} 0 & \phi_6 & 0 & \phi_4 & 0 & 0 \\ \phi_7 & 0 & \phi_2 & 0 & \phi_{13} & \phi_{14} \\ 0 & \phi_1 & 0 & \widetilde{\phi}^3 & 0 & 0 \\ 0 & \widetilde{\phi}^3 & 0 & \widetilde{\phi}^5 & 0 & 0 \\ \widetilde{\phi}^3 & 0 & \phi_{11} & 0 & \phi_8 & \widetilde{\phi}^3 \\ \widetilde{\phi}^3 & 0 & \phi_{12} & 0 & \widetilde{\phi}^3 & \phi_8 \end{pmatrix} . \tag{23.14}$$

Here we omit coefficients, which depend on the three Kähler moduli T_i and complex structure moduli U_i. Clearly, this mass matrix has rank five, such that there is one massless Higgs pair

$$h_u = a_1 \bar{h}_1 + a_2 \bar{h}_3 + a_3 \bar{h}_4 , \tag{23.15a}$$

$$h_d = b_1 h_1 + b_2 h_3 + b_3 h_5 + b_4 h_6 \tag{23.15b}$$

with a_i and b_j denoting coefficients. The $\bar{\delta} - \delta$ mass matrix is

$$\mathscr{M}_\delta = \begin{pmatrix} \widetilde{\phi}^5 & 0 & 0 \\ 0 & \phi_8 & \widetilde{\phi}^3 \\ 0 & \widetilde{\phi}^3 & \phi_8 \end{pmatrix} . \tag{23.16}$$

Hence, the matrix has full rank and all exotics decouple. Note that the block structure of $\mathscr{M}_\delta$ is not a coincidence but a consequence of the fact that δ_2/δ_3 and $\bar{\delta}_2/\bar{\delta}_3$ form D_4 doublets (see below). Altogether we see that all exotics with Higgs quantum numbers, and all but one pair of exotic triplets, decouple at the linear level in the $\widetilde{\phi}^{(i)}$ fields. This leads to the expectation that all but one pair of exotics get mass of the order of the GUT (or compactification) scale M_{GUT} while one pair of triplets might be somewhat lighter. We also note that the presence of colored states somewhat below M_{GUT} can give a better fit to MSSM gauge coupling unification (cf. [439]). However, a crucial property of the δ- and $\bar{\delta}$ triplets is that, due to the $\mathbb{Z}_4^R$ symmetry, they do not mediate dimension five proton decay.

Effective Yukawa Couplings

The effective Yukawa couplings are defined by

$$\mathscr{W}_Y = \sum_{i=1,3,4} \left[(Y_u^{(i)})^{fg}\, q_f\, \bar{u}_g\, \bar{h}_i \right] + \sum_{i=1,3,5,6} \left[(Y_d^{(i)})^{fg}\, q_f\, \bar{d}_g\, h_i + (Y_e^{(i)})^{fg}\, \ell_f\, \bar{e}_g\, h_i \right] . \tag{23.17}$$

The Yukawa coupling structures are

$$Y_u^{(1)} = \begin{pmatrix} \widetilde{\phi}^2 & \widetilde{\phi}^4 & \widetilde{\phi}^6 \\ \widetilde{\phi}^4 & \widetilde{\phi}^2 & \widetilde{\phi}^6 \\ \widetilde{\phi}^6 & \widetilde{\phi}^6 & 1 \end{pmatrix}, \quad Y_u^{(3)} = \begin{pmatrix} 1 & \widetilde{\phi}^6 & \widetilde{\phi}^4 \\ \widetilde{\phi}^6 & 1 & \widetilde{\phi}^4 \\ \widetilde{\phi}^4 & \widetilde{\phi}^4 & \widetilde{\phi}^2 \end{pmatrix}, \tag{23.18a}$$

$$Y_e^{(5)} = (Y_d^{(5)})^T = \begin{pmatrix} \widetilde{\phi}^6 & \widetilde{\phi}^6 & \widetilde{\phi}^6 \\ \widetilde{\phi}^6 & \widetilde{\phi}^6 & 1 \\ \widetilde{\phi}^6 & 1 & \widetilde{\phi}^4 \end{pmatrix}, \tag{23.18b}$$

$$Y_e^{(6)} = (Y_d^{(6)})^T = \begin{pmatrix} \widetilde{\phi}^6 & \widetilde{\phi}^6 & 1 \\ \widetilde{\phi}^6 & \widetilde{\phi}^6 & \widetilde{\phi}^6 \\ 1 & \widetilde{\phi}^6 & \widetilde{\phi}^4 \end{pmatrix}. \tag{23.18c}$$

Y_d and Y_e coincide at tree-level, i.e. they exhibit SU(5) GUT relations, originating from the non-local GUT breaking due to the freely acting Wilson line. There are additional contributions to Y_u from couplings to $\bar{h}_4$ and to Y_e/Y_d from couplings to $h_{1,3}$ which can be neglected if the VEVs of the $\widetilde{\phi}^{(i)}$ fields are small.

Because of the localization of the matter fields, we expect the renormalizable (1,3) and (3,1) entries in $Y_e^{(6)}$ to be exponentially suppressed.

Gauge-Top Unification

The $(3, 3)$ entry of Y_u is related to the gauge coupling. More precisely, in an orbifold GUT limit in which the first $\mathbb{Z}_2$ orbifold plane is larger than the other dimensions there is an SU(6) bulk gauge symmetry, and the ingredients of the top Yukawa coupling h_u (i.e. the fields $\bar{h}_{1,3,4}$), $\bar{u}_3$ and q_3 are bulk fields of this plane, i.e. hypermultiplets in the $N = 2$ supersymmetric description. As discussed in [440], this implies that the top Yukawa coupling y_t and the unified gauge coupling g coincide at tree-level. Moreover, localization effects in the two larger dimensions [455] will lead to a slight reduction of the prediction of y_t at the high scale such that realistic top masses can be obtained.

D_4 Flavor Symmetry

The block structure of the Yukawa matrices is not a coincidence but a consequence of a D_4 flavor symmetry [436], related to the vanishing Wilson line in the e_1 direction, $W_1 = 0$ (cf. e.g. [384]). The first two generations transform as a D_4 doublet, while the third generation is a D_4 singlet.

Neutrino Masses

In our model we have 11 neutrinos, i.e. SM singlets whose charges are odd under $\mathbb{Z}_4^R$ meaning that they have odd $\mathbb{Z}_2^{\mathscr{M}}$ charge, where $\mathbb{Z}_2^{\mathscr{M}}$ is the matter parity subgroup of $\mathbb{Z}_4^R$. Their mass matrix has rank 11 at the perturbative level. The neutrino Yukawa coupling is a 3×11 matrix and has full rank. Hence the neutrino see-saw mechanism with many neutrinos [456] is at work.

Proton Decay Operators

The $\mathbb{Z}_4^R$ symmetry forbids all dimension four and five proton decay operators at the perturbative level [353]. In addition, the non-anomalous matter parity subgroup $\mathbb{Z}_2^{\mathscr{M}}$ forbids all dimension four operators also non-perturbatively. The dimension five operators like $q\, q\, q\, \ell$ are generated non-pertubatively, as we will discuss below.

Non-perturbative Violation of $\mathbb{Z}_4^R$

Once we include the terms that are only forbidden by the $\mathbb{Z}_4^R$ symmetry, we obtain further couplings. An example for such an additional term is the dimension five proton decay operator,

$$\mathscr{W}_{np} \supset q_1\, q_1\, q_2\, \ell_1\, \mathrm{e}^{-a\,S}\,(x_4\bar{x}_5 + x_5\bar{x}_4) \left[\begin{pmatrix}\phi_{11}\\ \phi_{12}\end{pmatrix} \cdot \begin{pmatrix}\phi_{11}\\ \phi_{12}\end{pmatrix}\right]^3 \phi_4\, \phi_7^2 \left[\begin{pmatrix}\phi_9\\ \phi_{10}\end{pmatrix} \cdot \begin{pmatrix}\phi_9\\ \phi_{10}\end{pmatrix}\right] \tag{23.19}$$

where we suppressed coefficients. The bracket structure between the ϕ_{11}/ϕ_{12} and ϕ_9/ϕ_{10} is a consequence of the non-Abelian D_4 symmetry, where these fields transform as a doublet. The dot '$\cdot$' indicates the standard scalar product. Note that there are invariants with more than two D_4 charged fields which cannot be written in terms of a scalar product. Further, S is the dilaton and the coefficient $a = 8\pi^2$ in $\mathrm{e}^{-a\,S}$ is such that $\mathrm{e}^{-a\,S}$ has positive anomalous charge with respect to the normalized generator of the 'anomalous' U(1). This generator is chosen such that it is the gauge embedding of the anomalous space group element[2] (cf. Eq. (23.28)),

$$\mathrm{t}_{\mathrm{anom}} = W_3 + \mathrm{e8} \times \mathrm{e8 \ lattice\ vectors}\,. \tag{23.20}$$

The discrete Green-Schwarz mechanism is discussed in detail in [364].

[2]See [390] for the discussion in a more general context. Note that we can always bring the anomalous space group element to the form $(\theta^k\, \omega^\ell, 0)$ by redefining the model input appropriately. This amounts to a redefinition of the 'origin' of the orbifold.

Solution to the μ Problem

The $\mathbb{Z}_4^R$ anomaly has important consequences for the MSSM μ problem. The μ term is forbidden perturbatively by $\mathbb{Z}_4^R$, however, it appears at the non-perturbative level. Further, this model shares with the mini-landscape models the property that any allowed superpotential term can serve as an effective μ term (cf. the discussion in [371]). This fact can be seen from higher-dimensional gauge invariance [457]. Therefore, the (non-perturbative) μ term is of the order of the gravitino mass,

$$\mu \sim \langle \mathscr{W} \rangle \sim m_{3/2} \tag{23.21}$$

in Planck units. If some 'hidden' sector dynamics induces a non-trivial $\langle \mathscr{W} \rangle$, the μ problem is solved.

In this model, there is only a 'toy' hidden sector with an unbroken SU(2) gauge group and one pair of massless doublets whose mass term is prohibited by $\mathbb{Z}_4^R$. This sector has the structure discussed by Affleck, Dine and Seiberg (ADS) [458]. It turns out that the ADS superpotential is $\mathbb{Z}_4^R$ covariant. However, the hidden gauge group is probably too small for generating a realistic scale of supersymmetry breakdown. Yet there are alternative ways, such as the one described in [371], for generating a hierarchically small $\langle \mathscr{W} \rangle$.

Origin of $\mathbb{Z}_4^R$

In the orbifold CFT description the $\mathbb{Z}_4^R$ originates from the so-called H-momentum selection rules [430] (see also [383, 414]). These selection rules appear as discrete R symmetries in the effective field theory description of the model. It should be stressed that in large parts of the literature the order of these symmetries was given in an unfortunate way. For instance, the $\mathbb{Z}_2$ orbifold plane was said to lead to a $\mathbb{Z}_2^R$ symmetry, but it turned out that there are states with half-integer charges. It is more appropriate to call this symmetry $\mathbb{Z}_4^R$, and to deal with integer charges only. In the present model we have three $\mathbb{Z}_4^R$ symmetries at the orbifold point, stemming from the three $\mathbb{Z}_2$ orbifold planes.

H-momentum corresponds to angular momentum in the compact space; therefore the discrete R symmetries can be thought of as discrete remnants of the Lorentz symmetry of internal dimensions. That is to say that the orbifold compactification breaks the Lorentz group of the tangent space to a discrete subgroup. In this study we content ourselves with the understanding that these symmetries appear in the CFT governing the correlators to which we match the couplings of our effective field theory. The precise geometric interpretation of this symmetry in field theory will be discussed elsewhere.

Table 23.2 The states with their quantum number w.r.t. the SM and the hidden sector

Label	q_i	$\bar{u}_i$	$\bar{D}_i$	D_i	L_i	$\bar{L}_i$	$\bar{e}_i$	x_i	$\bar{x}_i$	y_i	z_i
#	3	3	6	3	9	6	3	5	5	6	6
$\mathrm{SU}(3)_C$	**3**	$\overline{\mathbf{3}}$	$\overline{\mathbf{3}}$	**3**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
$\mathrm{SU}(2)_\mathrm{L}$	**2**	**1**	**1**	**1**	**2**	**2**	**1**	**1**	**1**	**1**	**1**
$U(1)_Y$	$\frac{1}{3}$	-$\frac{4}{3}$	$\frac{2}{3}$	-$\frac{2}{3}$	-1	1	2	0	0	0	0
SU(3)	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**3**	$\overline{\mathbf{3}}$	**1**	**1**
SU(2)	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**2**	**1**
SU(2)	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**2**

The actual $\mathbb{Z}_4^R$ charges of $[\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(2)]_\mathrm{hid}$ invariant expressions in this model are given by

$$q_{\mathbb{Z}_4^R} \;=\; q_X + R_2 + 2n_3 \;, \tag{23.22}$$

where q_X is the U(1) charge generated by

$$\mathsf{t}_X \;=\; (4, 0, 10, -10, -10, -10, -10, -10)\;(-10, 0, 5, 5, -5, 15, -10, 0)\;, \tag{23.23}$$

R_2 denotes the R charge with respect to the second orbifold plane and n_3 is the localization quantum number in the third torus. The relevant quantum numbers are given in Table 23.2. The expression (23.22) for $q_{\mathbb{Z}_4^R}$ is not unique, there are 17 linear combinations of U(1) charges and discrete quantum numbers which can be used to rewrite the formula without changing the $\mathbb{Z}_4^R$ charges. Also the U(1) factors contained in $[\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{SU}(2)]_\mathrm{hid}$ can be used to redefine t_X. We refrain from spelling this out as we find it more convenient to work with invariant monomials (cf. the discussion in Appendix D [276]). It is straightforward to see that all monomials we switch on have R charge 0.

23.2 Summary

We have re-emphasized the important role of discrete symmetries in string model building. As an application, we discussed an explicit string model which exhibits MSSM vacua with a $\mathbb{Z}_4^R$ symmetry, which has recently been shown to be the unique symmetry for the MSSM that forbids the μ term at the perturbative level, allows Yukawa couplings and neutrino masses, and commutes with SO(10). This $\mathbb{Z}_4^R$ has a couple of appealing features. First, the μ term and dangerous dimension five proton decay operators are forbidden at the perturbative level and appear only through (highly suppressed) non-perturbative effects. Second, at the perturbative level, the expectation value of the superpotential is zero; a non-trivial expectation value is generated by non-perturbative effects. These two points imply that μ is of the order of the gravitino mass $m_{3/2}$, which is set by the expectation value of the superpotential (in Planck units).

The model is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold compactification of the $E_8 \times E_8$ heterotic string. We discussed how to search for field configurations which preserve $\mathbb{Z}_4^R$ and how to find supersymmetric vacua within such configurations. The Hilbert basis method allows one to construct a basis for all gauge invariant holomorphic monomials, and therefore to survey the possibilities of satisfying the D-term constraints. As we have seen, in the case of residual R symmetries it may in principle happen that the F-term equations over-constrain the system. This is not the case in the present model, i.e. there are supersymmetric vacua with the exact MSSM spectrum and a residual $\mathbb{Z}_4^R$ symmetry. Let us highlight the features of the model:

- exact MSSM spectrum, i.e. no exotics;
- almost all singlet fields/moduli are fixed in a supersymmetric way;
- non-local GUT breaking, i.e. the model is consistent with MSSM precision gauge unification;
- dimension four proton decay operators are completely absent as $\mathbb{Z}_4^R$ contains the usual matter parity as a subgroup;
- dimension five proton decay operators only appear at the non-perturbative level and are completely harmless;
- the gauge and top-Yukawa couplings coincide at tree level;
- see-saw suppressed neutrino masses;
- μ is related to the vacuum expectation value of the superpotential and therefore of the order of the gravitino mass;
- there is an SU(5) GUT relation between the τ and bottom masses.

There are also some drawbacks: first, there are SU(5) relations for the light generations and, second, the hidden sector gauge group is only SU(2) and therefore probably too small for explaining an appropriate scale of dynamical supersymmetry breaking. In addition, it may not be possible to blowup all the fixed points, in order to obtain a smooth supergravity limit [459]. Once again, as in the case of the "mini-landscape" models discussed earlier in Chap. 21, this is not necessarily a problem for the string model.

There is however one other major point in favor of this orbifold construction. The model has an orbifold GUT symmetry which is broken to the Standard Model via a non-local Wilson line. This is in contrast to the "mini-landscape" models where the orbifold GUT breaking was local. As discussed earlier in Chap. 18, with non-local GUT breaking the compactification scale is identified with the 4D GUT scale. The three Standard Model gauge couplings necessarily unify at this scale.

23.3 Details of the Model

In Table 23.3 we list the full spectrum. In addition to the states shown there, the spectrum contains the following (untwisted) moduli: the dilaton S, three Kähler moduli T_i and three complex structure moduli U_i.

The orbifold model is defined by a torus lattice that is spanned by six orthogonal vectors e_α, $\alpha = 1, \ldots, 6$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ twist vectors $v_1 = (0, 1/2, -1/2, 0)$ and $v_2 = (-1/2, 0, 1/2, 0)$, and the associated shifts

$$V_1 = \left(-\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0, 0\right)(0, 0, 0, 0, 0, 0, 0, 0)\ , \tag{23.24a}$$

$$V_2 = \left(0, \frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0\right)(0, 0, 0, 0, 0, 0, 0, 0)\ , \tag{23.24b}$$

and the six discrete Wilson lines

$$\begin{aligned}
W_1 &= \left(0^8\right)\left(0^8\right)\ , \qquad (23.25)\\
W_3 &= \left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)\left(0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1\right)\ ,\\
W_5 &= \left(-\frac{7}{4}, \frac{7}{4}, -\frac{1}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}\right)\left(-\frac{3}{4}, \frac{5}{4}, -\frac{5}{4}, -\frac{5}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}, \frac{5}{4}\right)\ ,\\
W_6 &= \left(\frac{3}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)\left(-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}, -\frac{3}{2}, \frac{3}{2}\right)\ ,\\
W_2 &= W_4 = W_6\ ,
\end{aligned}$$

corresponding to the six torus directions e_α. Additionally, we divide out the $\mathbb{Z}_2$ symmetry corresponding to

$$\tau = \frac{1}{2}(e_2 + e_4 + e_6) \tag{23.26}$$

with a gauge embedding denoted by W (the freely acting Wilson line) where

$$W = \frac{1}{2}(W_2 + W_4 + W_6) = \frac{3}{2}W_2\ . \tag{23.27}$$

The anomalous space group element reads

$$g_{\text{anom}} = (k, \ell; n_1, n_2, n_3, n_4, n_5, n_6) = (0, 0; 0, 0, 1, 0, 0, 0)\ , \tag{23.28}$$

Table 23.3 Spectrum of the model at the orbifold point

Sector	Irrep	q_{anom}	q_Y	q_X	q_1	q_2	q_3	q_4	q_5	q_6	$q_{\mathbb{Z}_4^R}$	(g)	(c)
U_1	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	4	0	−4	72	−88	356	188	−444	60	−4	N_1	ϕ_1
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	−4	$-\frac{1}{2}$	0	−4	0	8	0	0	0	0	$\bar{L}_1$	$\bar{h}_1$
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	4	$\frac{1}{2}$	0	4	0	−8	0	0	0	0	L_1	h_1
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−4	0	4	−72	88	−356	−188	444	−60	4	N_2	ϕ_2
U_2	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	6	$\frac{1}{2}$	−14	10	−14	62	30	−70	10	−12	L_3	h_3
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	0	10	66	−74	286	158	−374	50	12	N_5	ϕ_6
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−2	0	−10	−66	74	−286	−158	374	−50	−8	N_6	ϕ_7
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	−6	$-\frac{1}{2}$	14	−10	14	−62	−30	70	−10	16	$\bar{L}_3$	$\bar{h}_3$
U_3	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	0	−14	6	−14	70	30	−70	10	−14	N_3	$\bar{\phi}_4$
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	−2	$-\frac{1}{2}$	10	62	−74	294	158	−374	50	10	$\bar{L}_2$	$\bar{h}_2$
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	2	$\frac{1}{2}$	−10	−62	74	−294	−158	374	−50	−10	L_2	h_2
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−2	0	14	−6	14	−70	−30	70	−10	14	N_4	$\bar{\phi}_7$
$T_{(1,0)}^{(*,*,0,0,0,0)}$	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{1}{3}$	20	2	−2	12	0	0	0	21	$\bar{D}_1$	$\bar{d}_3$
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{2}$	20	2	−2	12	0	0	0	21	L_4	ℓ_3
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	4	0	−20	2	2	−20	0	0	0	−19	N_7	n_9
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	−1	0	2	0	−4	0	0	0	1	$\bar{E}_1$	$\bar{e}_3$
	$(\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	2	$-\frac{1}{6}$	0	2	0	−4	0	0	0	1	Q_1	q_3
	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	$\frac{2}{3}$	0	2	0	−4	0	0	0	1	$\bar{U}_1$	$\bar{u}_3$
$T_{(1,0)}^{(*,*,0,0,1,0)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	7	24	−31	140	60	−144	20	8	N_8	ϕ_8
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	4	0	−11	48	−57	216	128	−300	40	−10	N_9	$\bar{\phi}_{12}$
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	4	$-\frac{1}{2}$	−1	−14	15	−62	−30	74	−10	0	$\bar{L}_4$	$\bar{h}_4$
	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	4	$\frac{1}{3}$	−1	−14	15	−62	−30	74	−10	0	D_1	δ_1
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−8	0	1	10	−15	70	30	−74	10	2	N_{10}	$\bar{\phi}_1$

(continued)

Table 23.3 (continued)

Sector	Irrep	q_{anom}	q_Y	q_X	q_1	q_2	q_3	q_4	q_5	q_6	$q_{\mathbb{Z}_4^R}$	(g)	(c)
$T_{(1,0)}^{(*,*,0,0,1,1)}$	$(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})$	4	0	−16	−14	16	−62	−34	74	−10	−15	N_{11}	x_1
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	4	0	4	−14	16	−70	−26	66	−10	5	N_{12}	n_1
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$	3	0	−11	−14	16	−70	−30	74	−8	−10	N_{13}	z_1
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	3	0	−21	−14	16	−70	−30	82	−12	−20	N_{14}	y_1
$T_{(1,0)}^{(*,*,1,0,0,0)}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{3}},\mathbf{1},\mathbf{1})$	2	0	−12	36	−44	182	90	−218	30	−9	N_{15}	$\bar{x}_1$
	$(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})$	2	0	8	36	−44	174	98	−226	30	11	N_{16}	x_2
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	−2	36	−44	190	90	−226	30	1	N_{17}	n_2
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	4	0	−2	36	−44	166	98	−218	30	1	N_{18}	n_3
$T_{(1,0)}^{(*,*,1,0,0,1)}$	$(\mathbf{1},\mathbf{1},\overline{\mathbf{3}},\mathbf{1},\mathbf{1})$	2	0	3	−26	29	−104	−64	148	−20	6	N_{19}	$\bar{x}_2$
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	2	0	−17	−26	29	−96	−72	156	−20	−14	N_{20}	$\bar{\phi}_2$
	$(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})$	4	0	−7	−26	29	−112	−64	156	−20	−4	N_{21}	x_3
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	4	0	13	−26	29	−120	−56	148	−20	16	N_{22}	ϕ_3
$T_{(1,0)}^{(*,*,1,0,1,0)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	4	0	−11	−10	15	−70	−30	82	−10	−8	N_{23}	ϕ_4
	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{1}{3}$	11	14	−15	62	30	−82	10	14	$\bar{D}_2$	$\bar{\delta}_1$
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{2}$	11	14	−15	62	30	−82	10	14	L_5	h_4
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	21	−48	57	−216	−128	292	−40	24	N_{24}	ϕ_5
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	−4	0	−17	−24	31	−140	−60	152	−20	−14	N_{25}	$\bar{\phi}_3$
$T_{(1,0)}^{(*,*,1,0,1,1)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	3	0	−11	−14	16	−70	−30	74	−8	−8	N_{26}	y_2
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$	3	0	−21	−14	16	−70	−30	82	−12	−18	N_{27}	z_2
	$(\mathbf{1},\mathbf{1},\overline{\mathbf{3}},\mathbf{1},\mathbf{1})$	0	0	26	14	−16	62	34	−82	10	29	N_{28}	$\bar{x}_3$
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	6	14	−16	70	26	−74	10	9	N_{29}	n_4
$T_{(0,1)}^{(n_1,0,*,*,0,0)}$	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	1	$-\frac{1}{3}$	13	5	−9	47	15	−35	5	13	$\bar{D}_3, \bar{D}_4$	$\bar{d}_2, \bar{d}_1$
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	1	$\frac{1}{2}$	13	5	−9	47	15	−35	5	13	L_6, L_7	ℓ_2, ℓ_1
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	5	0	−27	5	−5	15	15	−35	5	−27	N_{30}, N_{36}	n_5, n_6

(continued)

Table 23.3 (continued)

Sector	Irrep	q_{anom}	q_Y	q_X	q_1	q_2	q_3	q_4	q_5	q_6	$q_{\mathbb{Z}_4^R}$	(g)	(c)
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	3	-1	-7	5	-7	31	15	-35	5	-7	$\bar{E}_2, \bar{E}_3$	$\bar{e}_2, \bar{e}_1$
	$(\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	3	$-\frac{1}{6}$	-7	5	-7	31	15	-35	5	-7	Q_2, Q_3	q_2, q_1
	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	3	$\frac{2}{3}$	-7	5	-7	31	15	-35	5	-7	$\bar{U}_2, \bar{U}_3$	$\bar{u}_2, \bar{u}_1$
$T_{(0,1)}^{(n_1,0,*,*,1,0)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	-1	0	14	21	-24	105	45	-109	15	14	N_{31}, N_{37}	$\bar{\phi}_5, \bar{\phi}_6$
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	3	0	-4	45	-50	181	113	-265	35	-4	N_{32}, N_{38}	ϕ_9, ϕ_{10}
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	3	$-\frac{1}{2}$	6	-17	22	-97	-45	109	-15	6	$\bar{L}_5, \bar{L}_6$	$\bar{h}_5, \bar{h}_6$
	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	3	$\frac{1}{3}$	6	-17	22	-97	-45	109	-15	6	D_2, D_3	δ_2, δ_3
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	-9	0	8	7	-8	35	15	-39	5	8	N_{33}, N_{39}	ϕ_{11}, ϕ_{12}
$T_{(0,1)}^{(n_1,0,*,*,1,1)}$	$(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})$	3	0	-9	-17	23	-97	-49	109	-15	-9	N_{34}, N_{40}	x_4, x_5
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	3	0	11	-17	23	-105	-41	101	-15	11	N_{35}, N_{41}	$\bar{n}_1, \bar{n}_2$
$T_{(1,1)}^{(n_1,0,0,0,*,*)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	5	0	3	-55	65	-251	-143	339	-45	4	N_{42}, N_{51}	ϕ_{13}, ϕ_{14}
	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	-3	$-\frac{1}{3}$	-13	-7	9	-43	-15	35	-5	-12	$\bar{D}_5, \bar{D}_6$	$\bar{\delta}_2, \bar{\delta}_3$
	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	-3	$\frac{1}{2}$	-13	-7	9	-43	-15	35	-5	-12	L_8, L_9	h_5, h_6
$T_{(1,1)}^{(n_1,0,0,1,*,*)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	6	0	13	7	-8	35	15	-43	7	14	N_{43}, N_{52}	y_3, y_5
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$	6	0	3	7	-8	35	15	-35	3	4	N_{44}, N_{53}	z_3, z_5
	$(\mathbf{1},\mathbf{1},\overline{\mathbf{3}},\mathbf{1},\mathbf{1})$	5	0	8	7	-8	27	19	-35	5	9	N_{45}, N_{54}	$\bar{x}_4, \bar{x}_5$
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	5	0	-12	7	-8	35	11	-27	5	-11	N_{46}, N_{55}	n_7, n_8
$T_{(1,1)}^{(n_1,0,1,0,*,*)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	5	0	3	3	-7	35	15	-43	5	6	N_{47}, N_{56}	$\bar{\phi}_8, \bar{\phi}_{10}$
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	-3	0	-17	3	-7	35	15	-27	5	-14	N_{48}, N_{57}	$\bar{\phi}_9, \bar{\phi}_{11}$
$T_{(1,1)}^{(n_1,0,1,1,*,*)}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$	6	0	13	7	-8	35	15	-43	7	16	N_{49}, N_{58}	z_4, z_6
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	6	0	3	7	-8	35	15	-35	3	6	N_{50}, N_{59}	y_4, y_6

The last two columns list the (g)eneral and the (c)onfiguration labels. If there are two labels in one line, this corresponds to the twist parameter $n_1 = 0$ for the first label and $n_1 = 1$ for the second. The two states form a doublet under a D_4 symmetry. In this model the three $\mathbb{Z}_4^R$ charges (corresponding to the three $\mathbb{Z}_2$ orbifold planes) of the respective sectors read: $R(U_1) = (2,0,0)$, $R(U_2) = (0,2,0)$, $R(U_3) = (0,0,2)$, $R(T_{(1,0)}) = (0,1,1)$, $R(T_{(0,1)}) = (1,0,1)$ and $R(T_{(1,1)}) = (1,1,0)$. *Reprinted from Nuclear Physics B 847, R. Kappl, B. Petersen, S. Raby, M. Ratz, R. Schieren, and P.K.S. Vaudrevange, "String-derived MSSM vacua with residual R symmetries," Page 345, Copyright (2011), with permission from Elsevier* where the boundary conditions of twisted string are

$$X(\tau, \sigma + 2\pi) = \vartheta^k \, \omega^\ell \, X(\tau, \sigma) + n_\alpha e_\alpha \tag{23.29}$$

with ϑ and ω denoting the rotations corresponding to v_1 and v_2. The spectrum is given in Table 23.2. In addition there are 37 $G_{\mathrm{SM}} \times [\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(2)]_{\mathrm{hid}}$ singlets.

Chapter 24
SUSY Breaking and Moduli Stabilization

In string models all couplings, i.e. both gauge and Yukawa couplings, depend on the values of moduli. In addition, the size of the extra 6 dimensions also depend on the size of moduli. Finally, the value of the GUT symmetry breaking scale and proton decay rates depend on the size of the extra dimensions. In addition, in order to stabilize all the moduli one must necessarily spontaneously break supersymmetry. All in all this remaining task of string model building is to say the least, daunting. Most string models have of order 100 moduli. These include the dilaton, Kähler (or volume) moduli, complex structure moduli and chiral moduli. The mechanism for stabilizing these moduli are different in the different string theory limits. In orbifold constructions of the heterotic string one typically considers simplified models with few moduli such as the dilaton, one volume modulus and perhaps one chiral modulus. The construction one uses makes use of non-perturbative gaugino condensates, world sheet instantons and anomalous $U(1)$ D-terms [453, 460–467]. For example in [453] we summarized these different mechanisms and the consequences for soft SUSY breaking parameters. In [181] it was shown how to obtain "mirage mediation" in the heterotic string context. Finally in [276] we showed that in the case of a heterotic orbifold model with a $\mathbb{Z}_4^R$ symmetry one can stabilize most of the moduli in a supersymmetric vacuum at the string scale. All others must be stabilized via the aforementioned mechanisms. Moduli stabilization in smooth constructions of the heterotic string have been considered in [468].

In type II string constructions it was shown that all geometric moduli can be stabilized by gauge fluxes [469]. The consequences for soft SUSY breaking parameters was worked out in [178, 179, 470]. Finally moduli stabilization in M-theory on G_2 manifolds was considered in [471, 472].

In this chapter we focus on the problem of moduli stabilization and SUSY breaking in the context of heterotic orbifold models. The analysis is based on the discussion in [453]. In Sect. 24.1 we summarize the general structure of the Kähler and superpotential in heterotic orbifold models. The models have a perturbative superpotential satisfying modular invariance constraints, an anomalous $U(1)_A$ gauge

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symmetry with a dynamically generated Fayet-Illiopoulos D-term and a hidden QCD-like non-Abelian gauge sector generating a non-perturbative superpotential. In Sect. 24.2 we consider a simple model with a dilaton, S, one volume modulus, T, and three standard model singlets. The model has only one gaugino condensate, as is the case for the "benchmark models" of the "mini-landscape" [406]. We obtain a 'hybrid KKLT' kind of superpotential that behaves like a single-condensate for the dilaton S, but as a racetrack[1] for the T and, by extension, also for the U moduli; and an additional matter F term, driven by the cancelation of an anomalous $U(1)_A$ D-term, is the seed for successful up-lifting. Previous analyses in the literature have also used an anomalous $U(1)_A$ D-term in coordination with other perturbative or non-perturbative terms in the superpotential to accomplish SUSY breaking and up-lifting [460, 462–467, 476–479]. We conclude that a single gaugino condensate is sufficient to break supersymmetry, stabilize all the moduli and generate a de Sitter vacuum.

24.1 General Structure

In this section we consider the supergravity limit of heterotic orbifold models. However, we will refer to the "mini-landscape" models for definiteness. We discuss the general structure of the Kähler potential, $\mathcal{K}$, the superpotential, $\mathcal{W}$, and gauge kinetic function, f_a for generic heterotic orbifold models. The "mini-landscape" models are defined in terms of a $\mathbb{Z}_6$-II orbifold of the six internal dimensions of the ten dimensional heterotic string. The orbifold is described by a three dimensional "twist" vector v, which acts on the compact directions. We define the compact directions in terms of complex coordinates:

$$\begin{aligned} Z_1 &\equiv X_4 + iX_5, \\ Z_2 &\equiv X_6 + iX_7, \\ Z_3 &\equiv X_8 + iX_9. \end{aligned} \tag{24.1}$$

The twist is defined by the action $Z_i \to e^{2\pi i v_i} Z_i$ for $i = 1, 2, 3$, and for $\mathbb{Z}_6$-II we have $v = \frac{1}{6}(1, 2, -3)$ or a (60°, 120°, 180°) rotation about the first, second and third torus, respectively. This defines the first twisted sector. The second and fourth twisted sectors are defined by twist vectors $2v$ and $4v$, respectively. Note, the third torus is unaffected by this twist. In addition, for the third twisted sector, generated by the twist vector $3v$, the second torus is unaffected. Finally the fifth twisted sector, given by $5v$ contains the CP conjugate states from the first twisted sector. Twisted sectors with un-rotated tori contain $N = 2$ supersymmetric spectra. This has consequences for the non-perturbative superpotential discussed in Sect. 24.1.3.

[1] Racetrack models for SUSY breaking are discussed in [473–475].

Finally, these models have three bulk volume moduli, $T_i,\ i = 1, 2, 3$ and one bulk complex structure modulus, U, for the third torus. In general, there are $h_{(1,1)}$ volume moduli and $h_{(2,1)}$ complex structure moduli.

24.1.1 Anomalous $U(1)_A$ and Fayet-Illiopoulos D-Term

The orbifold limit of the heterotic string has one anomalous $U(1)_A$ symmetry. The dilaton superfield S, in fact, transforms non-trivially under this symmetry. Let $V_A,\ V_a$ be the gauge superfields with gauge covariant field strengths, W_A^α, W_a^α, of gauge groups, $U(1)_A,\ \mathscr{G}_a$, respectively. The Lagrangian in the global limit is given in terms of a Kähler potential [15, 154, 480–482]

$$\mathscr{K} = -\log(S + \overline{S} - \delta_{GS} V_A) + \sum_a (\overline{Q}_a e^{V_a + 2q_a V_A} Q_a + \overline{\tilde{Q}}_a e^{-V_a + 2\tilde{q}_a V_A} \tilde{Q}_a) \tag{24.2}$$

and a gauge kinetic superpotential

$$\mathscr{W} = \tfrac{1}{2}[\tfrac{S}{4}(\textstyle\sum_a k_a Tr W_a^\alpha W_{\alpha a} + k_A Tr W_A^\alpha W_{\alpha A}) + h.c.]. \tag{24.3}$$

Note $q_a,\ \tilde{q}_a$ are the $U(1)_A$ charges of the 'quark', Q_a, and 'anti-quark', $\tilde{Q}_a$, supermultiplets transforming under $\mathscr{G}_a$ and $k_a,\ k_A$ are the Kač-Moody levels of the respective group factors.

Under a $U(1)_A$ super-gauge transformation with parameter Λ, one has[2]

$$\begin{aligned} \delta_A V_A &= -i(\Lambda - \bar{\Lambda})/2, \\ \delta_A S &= -i\frac{\delta_{GS}}{2}\Lambda, \end{aligned} \tag{24.4}$$

and

$$\delta_A \Phi = i q_\Phi \Lambda \Phi \tag{24.5}$$

for any charged multiplet Φ. The combination

$$S + \overline{S} - \delta_{GS} V_A \tag{24.6}$$

is $U(1)_A$ invariant. δ_{GS} is the Green-Schwarz coefficient given by

$$\delta_{GS} = 4\frac{TrQ_A}{192\pi^2} = \frac{(q_a + \tilde{q}_a)N_{f_a}}{4\pi^2} \tag{24.7}$$

[2] Note, the variation of the dilaton provides the Green-Schwarz cancelation of the $U(1)_A$ anomaly.

where the middle term is for the $U(1)_A$-gravity anomaly and the last term is for the $U(1)_A \times (\mathcal{G}_a)^2$ mixed anomaly.

The existence of an anomalous $U(1)_A$ has several interesting consequences. Due to the form of the Kähler potential [Eq. (24.2)] we obtain a Fayet-Illiopoulos D-term given by

$$\xi_A = \frac{\delta_{GS}}{2(S+\overline{S})} = -\frac{1}{2}\delta_{GS}\,\partial_S\mathcal{K} \tag{24.8}$$

with the $U(1)_A$ D-term contribution to the scalar potential given by

$$V_D = \frac{1}{S+\overline{S}}\left(\sum_a X_a^A \partial_a \mathcal{K}\,\phi^a + \xi_A\right)^2 \tag{24.9}$$

where X_a^A are Killing vectors for $U(1)_A$. In addition, clearly the perturbative part of the superpotential must be $U(1)_A$ invariant. But moreover, it constrains the non-perturbative superpotential as well. In particular, if the dilaton appears in the exponent, the product $e^{q_\Phi S}\Phi^{\delta_{GS}/2}$ is, and must also be, $U(1)_A$ invariant.

24.1.2 Target Space Modular Invariance

In this section, we wish to present the modular dependence of the gauge kinetic function, the Kähler potential, and of the superpotential in as general a form as possible. Most studies in the past have worked with a universal T modulus, and neglected the effects of the U moduli altogether. Such a treatment is warranted, for example, in the $\mathbb{Z}_3$ orbifolds where there are no U moduli. If we want to work in the limit of a stringy orbifold GUT [383] which requires one of the T moduli to be much larger than the others, or in the $\mathbb{Z}_6$-II orbifolds, however, it is impossible to treat all of the T and U moduli on the same footing.

Consider the $SL(2,\mathbb{Z})$ modular transformations of T and U given by [483–494][3]

$$T \to \frac{aT - ib}{icT + d}, \quad ad - bc = 1, \quad a,b,c,d \in \mathbb{Z}, \tag{24.10}$$

and

$$\log\left(T+\bar{T}\right) \to \log\left(\frac{T+\bar{T}}{(icT+d)(-ic\bar{T}+d)}\right), \tag{24.11}$$

[3]For an excellent review with many references, see [495].

with a similar transformation for U. The Kähler potential for moduli to zeroth order is given by:

$$\begin{aligned}\mathscr{K} &= -\sum_{i=1}^{h_{(1,1)}} \log\left(T^i + \bar{T}^i\right) - \sum_{j=1}^{h_{(2,1)}} \log\left(U^j + \bar{U}^j\right) \\ &= -\sum_{i=1}^{3} \log\left(T^i + \bar{T}^i\right) - \log\left(U + \bar{U}\right)\end{aligned} \tag{24.12}$$

where the last term applies to the "mini-landscape" models, since in this case $h_{(1,1)} = 3$, $h_{(2,1)} = 1$. Under the modular group, the Kähler potential transforms as

$$\mathscr{K} \to \mathscr{K} + \sum_{i=1}^{h_{(1,1)}} \log|ic_i T^i + d_i|^2 + \sum_{j=1}^{h_{(2,1)}} \log|ic_j U^j + d_j|^2. \tag{24.13}$$

The scalar potential V is necessarily modular invariant. We have

$$V = e^{\mathscr{G}}\left(\mathscr{G}_I \mathscr{G}^{I\bar{J}} \mathscr{G}_{\bar{J}} - 3\right) \tag{24.14}$$

where $\mathscr{G} = \mathscr{K} + \log|\mathscr{W}|^2$. Hence for the scalar potential to be invariant under the modular transformations, the superpotential must also transform as follows:

$$\begin{aligned}\mathscr{W} &\to \prod_{i=1}^{h_{(1,1)}} \prod_{j=1}^{h_{(2,1)}} (ic_i T^i + d_i)^{-1} (ic_j U^j + d_j)^{-1} \mathscr{W}, \\ \bar{\mathscr{W}} &\to \prod_{i=1}^{h_{(1,1)}} \prod_{j=1}^{h_{(2,1)}} (-ic_i \bar{T}^i + d_i)^{-1} (-ic_j \bar{U}^j + d_j)^{-1} \bar{\mathscr{W}}.\end{aligned} \tag{24.15}$$

This can be guaranteed by appropriate powers of the Dedekind η function multiplying terms in the superpotential.[4] This is due to the fact that under a modular transformation, we have

$$\eta(T) \to (icT + d)^{1/2} \eta(T), \tag{24.16}$$

[4]These terms arise as a consequence of world-sheet instantons in a string calculation. In fact, world sheet instantons typically result in more general modular functions [488–494].

up to a phase, where

$$\eta(T) = \exp(-\pi T/12) \prod_{n=1}^{\infty} \left(1 - e^{-2\pi n T}\right). \tag{24.17}$$

The Kähler potential for matter fields is of the form

$$\mathscr{K} \supset \Phi_I \, \bar{\Phi}^I \prod_{i=1}^{h_{(1,1)}} \left(T_i + \bar{T}_i\right)^{-n_I^i} \prod_{j=1}^{h_{(2,1)}} \left(U_j + \bar{U}_j\right)^{-\ell_I^j} = \Phi_I \, \bar{\Phi}^I \, \kappa_I. \tag{24.18}$$

The

$$\kappa_I \equiv \prod_i \left(T_i + \bar{T}_i\right)^{-n_I^i} \prod_j \left(U_j + \bar{U}_j\right)^{-\ell_I^j} \tag{24.19}$$

are the Kähler metrics for the chiral multiplets, Φ_I, and n_I^i, ℓ_I^i are the so-called modular weights.

The transformation of both the matter fields and the superpotential under the modular group fixes the modular dependence of the interactions. A field in the superpotential transforms as

$$\Phi_I \to \Phi_I \prod_{i=1}^{h_{(1,1)}} \prod_{j=1}^{h_{(2,1)}} \left(i c_i T^i + d_i\right)^{-n_I^i} \left(i c_j U^j + d_j\right)^{-\ell_I^j}. \tag{24.20}$$

The modular weights n_I^i and ℓ_I^j [446, 496] depend on the localization of the matter fields on the orbifold. For states I in the ith untwisted sector, i.e. those states with internal momentum in the ith torus, we have $n_I^i = \ell_I^i = 1$, otherwise the weights are 0. For twisted sector states, we first define $\boldsymbol{\eta}(k)$, which is related to the twisted sector $k(= 1, \ldots, N-1)$ and the orbifold twist vector v by

$$\eta_i(k) \equiv k v_i \mod 1. \tag{24.21}$$

Further, we require

$$\sum_i \eta_i(k) \equiv 1. \tag{24.22}$$

Then the modular weight of a state in the kth twisted sector is given by

$$\begin{aligned} n_I^i &\equiv (1 - \eta^i(k)) + N^i - \bar{N}^i \quad \text{for } \eta_i(k) \neq 0 \\ n_I^i &\equiv N^i - \bar{N}^i \quad \text{for } \eta_i(k) = 0. \end{aligned} \tag{24.23}$$

The N^i ($\bar{N}^i$) are *integer* oscillator numbers for left-moving oscillators $\tilde{\alpha}^i$ ($\tilde{\bar{\alpha}}^{\bar{i}}$), respectively. Similarly,

$$\begin{aligned} \ell_I^i &\equiv (1-\eta^i(k)) - N^i + \bar{N}^i \quad \text{for } \eta_i(k) \neq 0 \\ \ell_I^i &\equiv \qquad -N^i + \bar{N}^i \qquad\qquad \text{for } \eta_i(k) = 0. \end{aligned} \tag{24.24}$$

In general, one can compute the superpotential to arbitrary order in powers of superfields by a straightforward application of the string selection rules [497–500]. One assumes that any term not forbidden by the string selection rules appears with order one coefficient. In practice, even this becomes intractable quickly, and we must cut off the procedure at some low, finite order. More detailed calculations of individual terms give coefficients dependent on volume moduli due to string world sheet instantons. In general the moduli dependence can be obtained using the constraint of target space modular invariance. Consider a superpotential term for the "mini-landscape" models, with three T moduli and one U modulus, of the form:

$$\mathscr{W}_3 = Y_{IJK} \Phi_I \Phi_J \Phi_K. \tag{24.25}$$

We assume that the fields $\Phi_{I,J,K}$ transform with modular weights $n^i_{I,J,K}$ and $\ell^3_{I,J,K}$ under $T_i,\ i = 1,2,3$ and U, respectively. Using the (net) transformation property of the superpotential, and the transformation property of $\eta(T)$ under the modular group, we have (for non-universal moduli):

$$Y_{IJK} \sim h_{IJK} \prod_{i=1}^{3} \eta(T_i)^{\gamma_{T_i}} \eta(U)^{\gamma_U}$$

where $\gamma_{T_i} = -2(1 - n_I^i - n_J^i - n_K^i),\ \gamma_U = -2(1 - \ell_I^3 - \ell_J^3 - \ell_K^3)$ and h_{IJK} are dimensionless constants.[5] This is easily generalized for higher order interaction terms in the superpotential. We see that the modular dependence of the superpotential is rarely symmetric under interchange of the T_i or U_i. Note, when minimizing the scalar potential we shall use the approximation $\eta(T)^{\gamma_T} \approx e^{-bT}$ with $b = \pi\gamma_T/12$. (Recall, at large T, we have $\log(\eta(T)) \approx -\pi T/12$.) This approximation misses the physics near the self-dual point in the potential, nevertheless, it is typically a good approximation.

As a final note, Wilson lines break the $SL(2,\mathbb{Z})$ modular group down to a subgroup [501]. This has the effect of an additional differentiation of the moduli as they appear in the superpotential. In particular, factors of $\eta(T_i)$ are replaced by factors of $\eta(NT_i)$ or $\eta(T_i/N)$ for Wilson lines in $\mathbb{Z}_N$. In summary, the different

[5]Note, the constants $\gamma_{T_i},\ \gamma_U$ can quite generally have either sign, depending upon the modular weights of the fields at the particular vertex.

modular dependence of twisted sector fields and the presence of Wilson lines leads quite generally to anisotropic orbifolds [502].

24.1.3 Gauge Kinetic Function and Sigma Model Anomaly

To one loop, the string-derived gauge kinetic function is given by Dixon et al. [503], Derendinger et al. [504], Ibanez et al. [505], Lust and Munoz [506], Ibanez and Lust [446], Kaplunovsky and Louis [507]

$$
\begin{aligned}
f_a(S,T) = k_a S &+ \frac{1}{8\pi^2}\sum_{i=1}^{h_{(1,1)}} \left(\alpha_a^i - k_a\delta_\sigma^i\right)\log\left(\eta(T^i)\right)^2 \\
&+ \frac{1}{8\pi^2}\sum_{j=1}^{h_{(2,1)}} \left(\alpha_a^j - k_a\delta_\sigma^j\right)\log\left(\eta(U^j)\right)^2
\end{aligned}
\tag{24.26}
$$

where k_a is the Kač-Moody level of the group, which we will normally take to be 1. The constants α_a^i are model dependent, and are defined as

$$
\alpha_a^i \equiv \ell(\mathrm{adj}) - \sum_{\mathrm{rep}_I} \ell_a(\mathrm{rep}_I)(1 + 2n_I^i).
$$

$\ell(\mathrm{adj})$ and $\ell_a(\mathrm{rep}_I)$ are the Dynkin indices[6] of the adjoint representation and of the matter representation I of the group $\mathscr{G}_a$, respectively [57] and n_I^i are modular weights. The δ_σ^i terms are necessary to cancel an anomaly in the underlying σ-model, which induces a transformation in the dilaton field under the modular group:

$$
S \to S + \frac{1}{8\pi^2}\sum_{i=1}^{h_{(1,1)}} \delta_\sigma^i \log\left(ic_i T_i + d_i\right) + \frac{1}{8\pi^2}\sum_{j=1}^{h_{(2,1)}} \delta_\sigma^i \log\left(ic_j U^j + d_j\right). \tag{24.27}
$$

It is important to note that the factor

$$
\left(\alpha_a^i - k_a\delta_\sigma^i\right) \equiv \frac{b_a^{(N=2)}(i)}{|D|/|D_i|} \tag{24.28}
$$

where $b_a^{(N=2)}(i)$ is the beta function coefficient for the ith torus. It is non-zero if and only if the k-th twisted sector has an effective $N = 2$ supersymmetry. Moreover this occurs only when, in the k-th twisted sector, the ith torus is not rotated. The factors

[6] The Dynkin index $\ell_a(\mathrm{rep}_I) \equiv T_R$ defined earlier in Eq. (5.38). Thus if T_a^I are the generators of the group G_a in the representation I, then we have $Tr(T_a^I T_b^I) = \ell_a(\mathrm{rep}_I)\delta_{ab}$.

$|D|$, $|D_i|$ are the degree of the twist group D and the little group D_i, which does not rotate the ith torus. For example, for the "mini-landscape" models with $D = \mathbb{Z}_6$-II we have $|D| = 6$ and $|D_2| = 2$, $|D_3| = 3$ since the little group keeping the second (third) torus fixed is $\mathbb{Z}_2$ ($\mathbb{Z}_3$). The first torus is rotated in all twisted sectors. Hence, the gauge kinetic function for the "mini-landscape" models is only a function of T_2 and T_3.

Taking into account the sigma model anomalies, the heterotic string Kähler potential has the following form (where we have included the loop corrections to the dilaton [503, 504])

$$\mathcal{K} = -\log\left(S + \bar{S} + \frac{1}{8\pi^2}\sum_{i=1}^{h_{(1,1)}} \delta_\sigma^i \log\left(T^i + \bar{T}^i\right) + \frac{1}{8\pi^2}\sum_{j=1}^{h_{(2,1)}} \delta_\sigma^j \log\left(U^j + \bar{U}^j\right)\right)$$
$$-\sum_{i=1}^{h_{(1,1)}} \log\left(T^i + \bar{T}^i\right) - \sum_{j=1}^{h_{(2,1)}} \log\left(U^j + \bar{U}^j\right). \tag{24.29}$$

The first line of Eq. (24.29) is modular invariant by itself, and one can redefine the dilaton, Y, such that

$$Y \equiv S + \bar{S} + \frac{1}{8\pi^2}\sum_{i=1}^{h_{(1,1)}} \delta_\sigma^i \log\left(T^i + \bar{T}^i\right) + \frac{1}{8\pi^2}\sum_{j=1}^{h_{(1,2)}} \delta_\sigma^j \log\left(U^j + \bar{U}^j\right), \tag{24.30}$$

where Y is invariant under the modular transformations.

24.1.4 Non-perturbative Superpotential

In all "mini-landscape" models [437], and most orbifold heterotic string constructions, there exists a hidden sector with non-Abelian gauge interactions and vector-like matter carrying hidden sector charge. In the "benchmark" models [406] the hidden sector gauge group is $SU(4)$ with chiral matter in the $4+\bar{4}$ representation.

In this section let us consider a generic hidden sector with gauge group $SU(N_1) \otimes SU(N_2) \otimes U(1)_A$, where '$A$' stands for anomalous.[7] There are N_{f_1} and N_{f_2} flavors of quarks Q_1 and Q_2 in the fundamental representation (along with anti-quarks $\tilde{Q}_1$ and $\tilde{Q}_2$, in the anti-fundamental representations), as well as two singlet fields, called ϕ and χ. The charge assignments are listed in Table 24.1. We assume the existence of two moduli, S and T, which enter the non-perturbative superpotential

[7] Note, for clarity, this is just a toy model which is not derived directly from any particular string model.

Table 24.1 Charge assignments for the fields in a generic hidden sector

	ϕ	χ	Q_1	Q_2	$\tilde{Q}_1$	$\tilde{Q}_2$
$U(1)_A$	-1	q_χ	q_1	q_2	$\tilde{q}_1$	$\tilde{q}_2$
SU(N_1)	1	1	$\square$	1	$\bar{\square}$	1
SU(N_2)	1	1	1	$\square$	1	$\bar{\square}$

Flavor indices are suppressed

through the gauge kinetic function, namely $f = f(S, T)$. The model also allows for T dependence in the Yukawa sector.

Non-perturbative effects generate a potential for the S and T moduli. Gaugino condensation will generate a scale Λ_{SQCD}, which is determined purely by the symmetries of the low energy theory:

$$\Lambda_a(S, T) = e^{-\frac{8\pi^2}{\beta_a} f_a(S,T)}, \tag{24.31}$$

where $\beta_a = 3N_a - N_{f_a}$ is the one loop beta function coefficient of the theory. At tree level $f_a(S, T) = S$, however, we include the possibility of threshold corrections which introduce a dependence on the T modulus [503, 504]. We also find that $U(1)_A$ and modular invariance together dictate a very specific form for the non-perturbative superpotential.

In the "mini-landscape" analysis the effective mass terms for the vector-like exotics were evaluated. They were given as a polynomial in products of chiral MSSM singlet fields [chiral moduli]. It was shown that all vector-like exotics obtain mass[8] when the chiral moduli obtain VEVs at supersymmetric points in moduli space. In our example let us, for simplicity, take couplings between the quarks and the field ϕ to be diagonal in flavor space. Mass terms of the form

$$\mathbb{M}_1(\phi, T) Q_1 \tilde{Q}_1 + \mathbb{M}_2(\phi, T) Q_2 \tilde{Q}_2 \tag{24.32}$$

are dynamically generated when ϕ receives a non-zero VEV, which we will discuss below. A key assumption is that those mass terms are larger than the scale of gaugino condensation, so that the quarks and anti-quarks may be consistently integrated out. If this can be accomplished, then one can work in the pure gauge limit [508].[9]

[8] In fact, one of the $SU(4)$ quark- anti-quark pairs remained massless in the two "benchmark" models.

[9] There is a check on the consistency of this approach: at the end of the day, after calculating the VEVs of the scalars, we can verify that the mass terms for the quarks are indeed of the correct magnitude.

Before we integrate out the meson fields ($\mathscr{M}_a \equiv Q_a\tilde{Q}_a,\ a = 1, 2$),[10] the non-perturbative superpotential (plus quark masses) for $N_{f_a} < N_a$ is of the form [458]

$$\mathscr{W}_{\rm NP} = \sum_{a=1,2}\left[\mathbb{M}_a(\phi,T)Q_a\tilde{Q}_a + (N_a - N_{f_a})\left(\frac{\Lambda_a^{3N_a-N_{f_a}}}{\det Q_a\tilde{Q}_a}\right)^{\frac{1}{N_a-N_{f_a}}}\right], \tag{24.33}$$

with $\mathbb{M}_a(\phi,T) = c_a e^{-b_a T}\phi^{q_a+\tilde{q}_a}$ where c_a is a constant. Note, given the charges for the fields in Table 24.1 and using Eqs. (24.4), (24.7) and (24.31), one sees that $\mathscr{W}_{\rm NP}$ is $U(1)_A$ invariant. The Kähler potential for the hidden sector is assumed to be of the form

$$\begin{aligned}\mathscr{K} &= -\log(S+\overline{S}) - 3\log(T+\overline{T}) + \alpha_\phi\overline{\phi}e^{-2V_A}\phi + \alpha_\chi\overline{\chi}e^{2q_\chi V_A}\chi \\ &\quad + \sum_{a=1,2}\alpha_a(\overline{Q}_a e^{V_a+2q_aV_A}Q_a + \overline{\tilde{Q}}_a e^{-V_a+2\tilde{q}_aV_A}\tilde{Q}_a)\end{aligned} \tag{24.34}$$

The quantities $\alpha_\phi, \alpha_\chi, \alpha_i$ are generally functions of the modulus T, where the precise functional dependence is fixed by the modular weights of the fields (see Sect. 24.1.2). V_i and V_A denote the vector superfields associated with the gauge groups $\mathscr{G}_i = SU(N_i)$ and $U(1)_A$.

The determinant of the quark mass matrix is given by

$$\det\mathbb{M}_a(\phi,T) = \left(c_a e^{-b_aT}\phi^{q_a+\tilde{q}_a}\right)^{N_{f_a}}. \tag{24.35}$$

We have taken the couplings between ϕ and the quarks to have exponential dependence on the T modulus, an ansatz which is justified by modular invariance (see Sect. 24.1.2). Inserting the meson equations of motion and Eq. (24.35) into Eq. (24.33), we have

$$\mathscr{W}_{\rm NP} = \sum_{a=1,2}\left[N_a\left(c_a e^{-b_aT}\phi^{q_a+\tilde{q}_a}\right)^{\frac{N_{f_a}}{N_a}}[\Lambda_a(S,T)]^{\frac{3N_a-N_{f_a}}{N_a}}\right].$$

Note that the transformation of the superpotential under the modular group in Eq. (24.15) also requires that the (non-perturbative) superpotential obey

$$\mathscr{W}_{\rm NP} \to \prod_{i=1}^{h_{(1,1)}}\prod_{j=1}^{h_{(2,1)}}(ic_iT^i + d_i)^{-1}(ic_jU^j + d_j)^{-1}\mathscr{W}_{\rm NP}. \tag{24.36}$$

[10] The meson field, $Q_a\tilde{Q}_a$ is assumed to be diagonal and proportional to the identity in flavor space. Thus not breaking the $SU(N_{f_a})$ flavor symmetry.

Because the non-perturbative lagrangian must be invariant under all of the symmetries of the underlying string theory, it must be that [475, 506, 509–512]:

$$\mathscr{W}_{\mathrm{NP}} \equiv A \times e^{-aS} \prod_{i=1}^{h_{(1,1)}} \prod_{j=1}^{h_{(2,1)}} \left(\eta(T^i)\right)^{-2+\frac{3}{4\pi^2\beta}\delta^i_\sigma} \left(\eta(U^j)\right)^{-2+\frac{3}{4\pi^2\beta}\delta^j_\sigma} \tag{24.37}$$

where $a \equiv \frac{24\pi^2}{\beta}$ and $\beta = 3\ell(\mathrm{adj}) - \sum_I \ell(\mathrm{rep}_I)$ is the one-loop beta function coefficient, and A is generally a function of the chiral matter fields appearing in $\mathbb{M}_a(\phi, T)$. This, coupled with the one loop gauge kinetic function in Eq. (24.26), gives the heterotic generalization of the Racetrack superpotential.

In the following Sect. 24.2, we construct a simple model using the qualitative features outlined in this section. This model is novel because it requires only one non-Abelian gauge group to stabilize moduli and give a de Sitter vacuum. We have also constructed two condensate models, however, the literature already contains several examples of the "racetrack" in regards to stabilization of S and T moduli. Moreover in the "mini-landscape" models, whose features we are seeking to reproduce, there are many examples of hidden sectors containing a single non-Abelian gauge group [437], while there are no examples with multiple hidden sectors.

24.2 Moduli Stabilization and Supersymmetry Breaking in the Bulk

In this section we construct a simple, generic heterotic orbifold model which captures many of the features discussed in Sect. 24.1. In particular, it is a single gaugino condensate model with the following fields—dilaton (S), modulus (T) and MSSM singlets (ϕ_1, ϕ_2, χ). The model has one anomalous $U(1)_A$ with the singlet charges given by ($q_{\phi_1} = -2, q_{\phi_2} = -9, q_\chi = 20$). The Kähler and superpotential are given by[11]

$$\mathscr{K} = -\log[S + \bar{S}] - 3\log[T + \overline{T}] + \overline{\phi}_1\phi_1 + \overline{\phi}_2\phi_2 + \overline{\chi}\chi \tag{24.38}$$

$$\mathscr{W} = e^{-bT}(w_0 + \chi(\phi_1^{10} + \lambda\phi_1\phi_2^2)) + A\,\phi_2^p\, e^{-aS-b_2T}. \tag{24.39}$$

[11]The coefficient A [Eq. (24.39)] is an implicit function of all other non-vanishing chiral singlet VEVs which would be necessary to satisfy the modular invariance constraints, i.e. $A = A(\langle\phi_I\rangle)$. If one re-scales the $U(1)_A$ charges, $q_{\phi_i}, q_\chi \to q_{\phi_i}/r, q_\chi/r$, then the $U(1)_A$ constraint is satisfied with $r = 15p$ (assuming no additional singlets in A). Otherwise we may let r and p be independent. This re-scaling does not affect our analysis, since the vacuum value of the ϕ_i, χ term in the superpotential vanishes.

In addition, there is an anomalous $U(1)_A$ D-term given by

$$D_A = 20\overline{\chi}\chi - 2\overline{\phi}_1\phi_1 - 9\overline{\phi}_2\phi_2 - \tfrac{1}{2}\delta_{GS}\,\partial_S\mathscr{K} \tag{24.40}$$

with $\delta_{GS} = \frac{(q+\tilde{q})N_f}{4\pi^2} = N_f/(4\pi^2)$.

In the absence of the non-perturbative term (with coefficient A) the theory has a supersymmetric minimum with $\langle\chi\rangle = \langle\phi_1\rangle = 0$ and $\langle\phi_2\rangle \neq 0$ and arbitrary. This property mirrors the situation in the "mini-landscape" models where supersymmetric vacua have been found in the limit that all non-perturbative effects are neglected. We have also added a constant $w_0 = w_0(\langle\phi_I\rangle)$ which is expected to be generated (in the "mini-landscape" models) at high order in the product of chiral moduli due to the explicit breaking of an accidental R symmetry which exists at lower orders [371].[12] The T dependence in the superpotential is designed to take into account, in a qualitative way, the modular invariance constraints of Sect. 24.1.2. We have included only one T modulus, assuming that the others can be stabilized near the self-dual point [509, 513]. Moreover, as argued earlier, the T_i and U moduli enter the superpotential in different ways (see Sect. 24.1.2). This leads to modular invariant solutions which are typically anisotropic [502].[13]

Note, that the structure, $\mathscr{W} \sim w_0 e^{-bT} + \phi_2\, e^{-aS-b_2T}$ gives us the crucial progress[14]:

1. a 'hybrid KKLT' kind of superpotential that behaves like a single-condensate for the dilaton S, but as a racetrack for the T and, by extension, also for the U moduli; and
2. an additional matter F_{ϕ_2} term driven by the cancelation of the anomalous $U(1)_A$ D-term seeds SUSY breaking with successful uplifting.

The constant b is fixed by modular invariance constraints. In general the two terms in the perturbative superpotential would have different T dependence. We have found solutions for this case as well. This is possible since the VEV of the χ term in the superpotential vanishes. The second term (proportional to A) represents the non-perturbative contribution of one gaugino condensate. The constants $a = 24\pi^2/\beta,\ b_2$ and p depend on the size of the gauge group, the number of flavors and the coefficient of the one-loop beta function for the effective $N = 2$ supersymmetry of the torus T. For the "mini-landscape" models, this would be either T_2 or T_3. Finally, the coefficient of the exponential factor of the dilaton S is taken to be $A\,\phi_2^p$. This represents the effective hidden sector quark mass term, which in this case is proportional to a power of the chiral singlet ϕ_2. In a more general case, it would be

[12] The fields entering w_0 have string scale mass.

[13] Note, we have chosen to keep the form of the Kähler potential for this single T modulus with the factor of 3, so as to maintain the approximate no-scale behavior.

[14] Note, the constants $b,\ b_2$ can have either sign. For the case with $b,\ b_2 > 0$ the superpotential for T is racetrack-like. However for $b,\ b_2 < 0$ the scalar potential for T diverges as T goes to zero or infinity and compactification is guaranteed [509, 513].

Table 24.2 Input values for the superpotential parameters for three different cases

Case	b	b_2	λ	R	p	r	A	w_0
1	$\pi/50$	$3\pi/2$	33	10	$2/5$	$15p$	160	8×10^{-15}
2	$8/125$	$3\pi/2$	0	5	$2/5$	$15p$	30	42×10^{-16}
3	$1/16$	$29\pi/20$	38	10	$2/5$	$15p$	90	6×10^{-15}
4	$-\pi/120$	$-\pi/40$	40	64	$2/3$	1	$1/10$	-5×10^{-15}
5	$-\pi/250$	$-\pi/100$	25	16	1	$10/3$	$7/5$	-7×10^{-15}

Case 2 has a vanishing one loop correction for ϕ_2

a polynomial in powers of chiral moduli.[15] The exponent p depends in general on the size of the gauge group, the number of flavors and the power that the field ϕ_2 appears in the effective quark mass term.

We have performed a numerical evaluation of the scalar potential with the following input parameters. We take hidden sector gauge group $SU(N)$ with $N = 5$, $N_f = 3$ and $a = 8\pi^2/N$.[16] For the other input values we have considered five different possibilities given in Table 24.2.[17] We find that supersymmetry breaking, moduli stabilization and up-lifting is a direct consequence of adding the non-perturbative superpotential term.

In our analysis we use the scalar potential V given by

$$V = e^K\Big(\sum_{i=1}^{5}\sum_{j=1}^{5}\big[F_{\Phi_i}\overline{F_{\Phi_j}}\,\mathcal{K}^{-1}_{i,j} - 3|W|^2\big]\Big) + \frac{D_A^2}{(S+\bar{S})} + \Delta V_{CW}[\Phi_i,\overline{\Phi}_i] \tag{24.41}$$

where $\Phi_{i,j} = \{S, T, \chi, \phi_1, \phi_2\}$ and $F_{\Phi_i} \equiv \partial_{\Phi_i}\mathcal{W} + (\partial_{\Phi_i}\mathcal{K})\mathcal{W}$. The first two terms are the tree level supergravity potential. The last term is a one loop correction which affects the vacuum energy and D term contribution.

The one loop Coleman-Weinberg potential is in general given by

$$\Delta V_{CW} = \frac{1}{32\pi^2}Str(M^2)\Lambda^2 + \frac{1}{64\pi^2}Str(M^4\log[\frac{M^2}{\Lambda^2}]) \tag{24.42}$$

[15]Holomorphic gauge invariant monomials span the moduli space of supersymmetric vacua. One such monomial is necessary to cancel the Fayet-Illiopoulos D-term (see Sect. 24.5).

[16]We have also found solutions for the case with $N = 4$, $N_f = 7$ which is closer to the "mini-landscape" benchmark models. Note, when $N_f > N$ we may still use the same formalism, since we assume that all the $Q, \tilde{Q}$s get mass much above the effective QCD scale.

[17]Note the parameter relation $r = 15p$ in Table 24.2 is derived using $U(1)_A$ invariance and the assumption that no other fields with non-vanishing $U(1)_A$ charge enter into the effective mass matrix for hidden sector quarks. We have also allowed for two cases where this relation is not satisfied.

with the mass matrix M given by $M = M(\Phi_i)$ and Λ is the relevant cut-off in the problem. We take $\Lambda = M_S \sim 10^{17}$ GeV.

We have not evaluated the full one loop correction. Instead we use the approximate formula

$$\Delta V_{CW}[\phi_2, \overline{\phi_2}] = \frac{\lambda^2 F_2^2 |\phi_2|^2}{8\pi^2} \left(\log[R(\lambda|\phi_2|^2)^2] + 3/2\right) + \mathcal{O}(\Lambda^2) \tag{24.43}$$

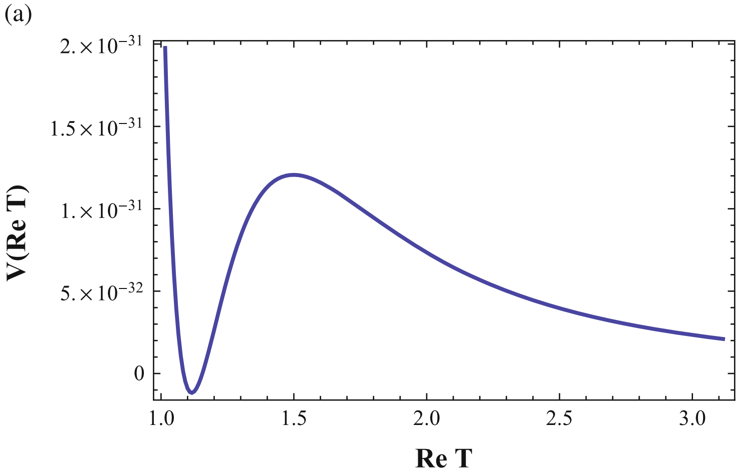
where $F_2 = \langle F_{\phi_2} \rangle$ is obtained self-consistently and all dimensionful quantities are expressed in Planck units. This one loop expression results from the χ, ϕ_1 contributions to the Coleman-Weinberg formula. The term quadratic in the cut-off is naturally proportional to the number of chiral multiplets in the theory and could be expected to contribute a small amount to the vacuum energy, of order a few percent times $m_{3/2}^2 M_{pl}^2$. We will discuss this contribution later, after finding the minima of the potential. Finally, note that the parameters λ, R in Table 24.2 might both be expected to be significantly greater than one when written in Planck units. This is because the scale of the effective higher dimensional operator with coefficient λ in Eq. (24.39) is most likely set by some value between M_{Pl} and M_{string} and the cut-off scale for the one loop calculation (which determines the constant R) is the string scale and not M_{Pl}.

In all cases we find a meta-stable minimum with all (except for two massless modes) fields massive of $\mathcal{O}$(TeV) or larger. Supersymmetry is broken at the minimum with values given in Table 24.3. Note $Re\, S \sim 2.2$ and $Re\, T$ ranges between 1.1 and 1.6. The moduli χ, ϕ_1 are stabilized at their global minima $\phi_1 = \chi = 0$ with $F_\chi = F_{\phi_1} = 0$ in all cases. The modulus $\sigma = \text{Im}\ S$ is stabilized at $\sigma \approx 1$ in the racetrack cases 1, 2 and 3. This value enforces a relative negative sign between the two terms dependent on $Re\, T$. We plot the scalar potential V in the $Re\, T$ direction

Table 24.3 The values for field VEVs and soft SUSY breaking parameters at the minimum of the scalar potential

	Case 1	Case 2	Case 3	Case 4	Case 5
$\langle s \rangle$	2.2	2.2	2.1	2.1	2.2
$\langle t \rangle$	1.2	1.1	1.6	1.1	1.1
$\langle \sigma \rangle$	1.0	1.0	1.0	0.0	0.0
$\langle \phi_2 \rangle$	0.08	0.08	0.08	0.03	0.06
F_S	2.8×10^{-16}	1.3×10^{-16}	2.7×10^{-16}	1.1×10^{-16}	8.0×10^{-17}
F_T	-8.7×10^{-15}	-5.1×10^{-15}	-5.0×10^{-15}	6.7×10^{-15}	9.1×10^{-15}
F_{ϕ_2}	-9.2×10^{-17}	-4.5×10^{-17}	-8.9×10^{-17}	1.3×10^{-15}	1.3×10^{-15}
D_A	4.4×10^{-31}	1.0×10^{-32}	5.9×10^{-31}	-3.8×10^{-31}	-4.8×10^{-32}
$D_A/m_{3/2}^2$	0.6	0.03	2.7	−0.7	−0.05
$V_0/(3m_{3/2}^2)$	−0.02	−0.01	−0.02	−0.03	−0.02
$m_{3/2}$ (TeV)	2.2	1.4	1.1	1.8	2.4

Note: $F_\Phi \equiv \partial_\Phi \mathcal{W} + (\partial_\Phi \mathcal{K})\mathcal{W}$



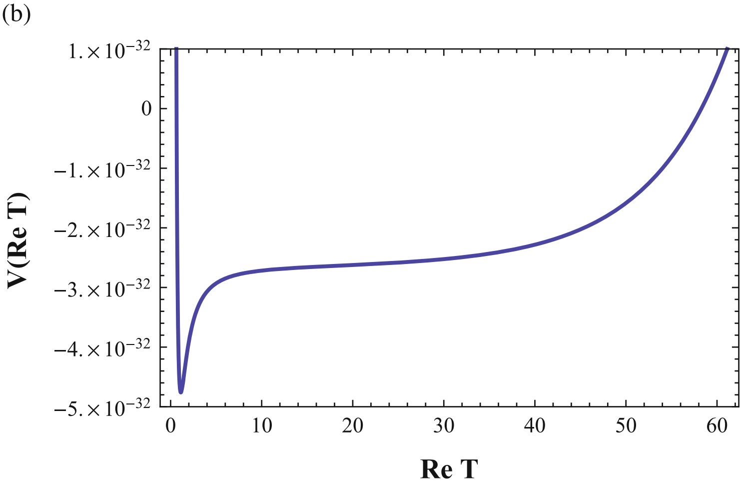


Fig. 24.1 As Re $T \to \infty$, the potential for $b_i > 0$ mimics a Racetrack, which can be seen from Eq. (24.39), for example. In the case where $b_i < 0$, however, the potential exhibits a different asymptotic behavior. As Re $T \to \infty$ the potential diverges, which means that theory is forced to be compactified [509, 513]. (**a**) The scalar potential in Case 2 for Re T, with $b_i > 0$. (**b**) The scalar potential in Case 4 for Re T, with $b_i < 0$

for case 2 ($b,\ b_2 > 0$) (Fig. 24.1a) and for case 4 ($b,\ b_2 < 0$) (Fig. 24.1b). Note the potential as a function of *Re S* is qualitatively the same for both cases (Fig. 24.2).

At the meta-stable minimum of the scalar potential we find a vacuum energy which is slightly negative, i.e. of order $(-0.03 \text{ to } -0.01) \times 3m_{3/2}^2 M_{Pl}^2$ (see Table 24.3). Note, however, one loop radiative corrections to the vacuum energy are of order $(N_T\ m_{3/2}^2 M_S^2/16\pi^2)$,where N_T is the total number of chiral multiplets [514] and we have assumed a cut-off at the string scale M_S. With typical values $N_T \sim \mathscr{O}(300)$ and $M_S/M_{Pl} \sim 0.1$, this can easily lift the vacuum energy the rest of the

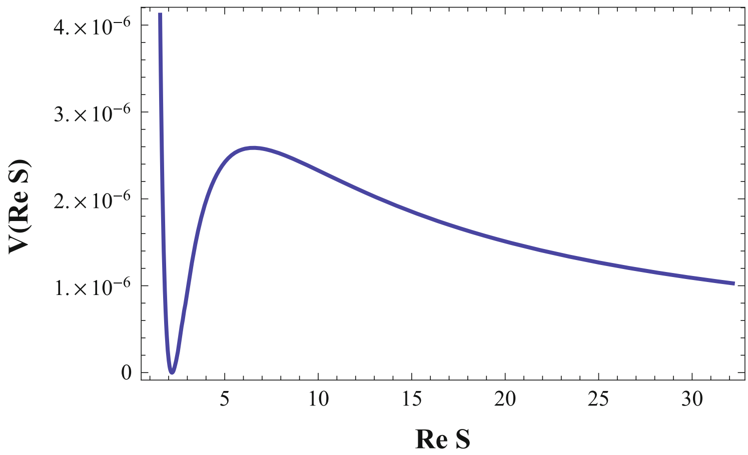


Fig. 24.2 The scalar potential in the *ReS* direction for Case 2

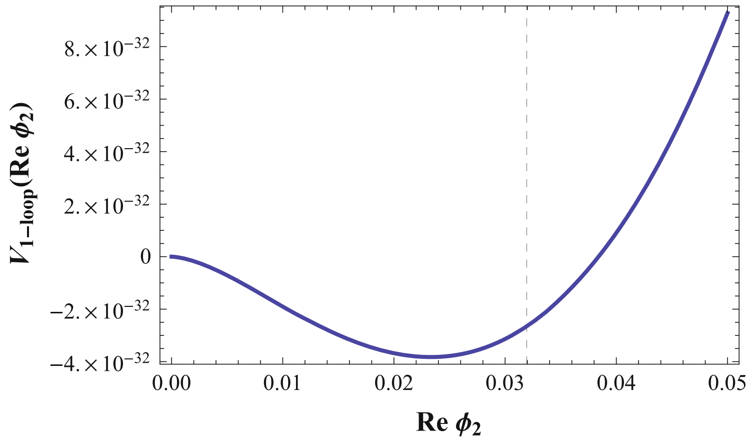


Fig. 24.3 The one loop Coleman-Weinberg potential (Case 4) for ϕ_2. The *dashed line* represents the VEV of ϕ_2 in the minimum of the *full* potential

way to give a small positive effective cosmological constant which is thus a metastable local dS minimum. Note that the constants λ, R have also been used to adjust the value of the cosmological constant as well as, and more importantly for LHC phenomenology, the value of D_A (see Fig. 24.3).

The two massless fields can be seen as the result of two $U(1)$ symmetries; the first is a $U(1)_R$ symmetry and the second is associated with the anomalous $U(1)_A$. The $U(1)_R$ is likely generic (but approximate), since even the "constant" superpotential term needed to obtain a small cosmological constant necessarily comes with $\eta(T)$ moduli dependence. Since we have approximated $\eta(T) \sim \exp(-\pi T/12)$ by the first term in the series expansion [Eq. (24.17)], the symmetry is exact. However higher order terms in the expansion necessarily break the $U(1)_R$ symmetry. The $U(1)_A$ symmetry is gauged.

One can express the fields S, T, and ϕ_2 in the following basis[18]:

$$\begin{aligned} S &\equiv s + i\sigma, \\ T &\equiv t + i\tau, \\ \phi_2 &\equiv \varphi_2\, e^{i\theta_2}. \end{aligned} \tag{24.44}$$

The transformation properties of the fields σ, τ and θ_2 under the two U(1)'s are given by

$$\begin{aligned} U(1)_R &: \begin{cases} \tau \to \tau + c \\ \sigma \to \sigma + \frac{-b_2+b}{a} c \end{cases}, \\ U(1)_A &: \begin{cases} \theta \to \theta - \frac{9}{r} c' \\ \sigma \to \sigma - \frac{9p}{a \cdot r} c' \end{cases}, \end{aligned} \tag{24.45}$$

where c, c' are arbitrary constants and for the definition of r see footnote 11. The corresponding Nambu-Goldstone (NG) bosons are given by

$$\begin{aligned} \chi^1_{\mathrm{NG}} &= \frac{a}{-b_2 + b}\sigma + \tau, \\ \chi^2_{\mathrm{NG}} &= \tilde{N}\left(-\sigma + \frac{-b_2 + b}{a}\tau\right) + \frac{1}{p}\theta_2, \end{aligned} \tag{24.46}$$

where $\tilde{N}$ is a normalization factor. One can then calculate the mass matrix in the $\sigma - \tau - \theta_2$ basis and find two zero eigenvalues (as expected) and one non-zero eigenvalue. The two NG modes, in all cases, can be shown to be linear combinations of the two eigenvectors of the two massless states. The $U(1)_A$ NG boson is eaten by the $U(1)_A$ gauge boson, while the $U(1)_R$ pseudo-NG boson remains as an "invisible axion" [515]. The $U(1)_R$ symmetry is non-perturbatively broken (by world-sheet instantons) at a scale of order

$$\langle e^{\mathcal{K}/2} \mathcal{W}\; e^{-\pi T} \rangle \approx m_{3/2} \langle e^{-\pi T} \rangle \sim 0.02\, m_{3/2} \tag{24.47}$$

in Planck units, resulting in an "axion" mass of order 800 GeV (for $m_{3/2} \sim 40$ TeV) and decay constant of order M_{PL}. Such a light pseudo-Nambu-Goldstone boson might contribute to a cosmological moduli problem. In addition, the heterotic orbifold models might very well have the standard invisible axion [516–519].

[18] The fields χ and ϕ_1 cannot be expressed in polar coordinates as they receive zero VEV, and cannot be canonically normalized in this basis.

24.3 SUSY Spectrum

Now that we understand how SUSY is broken, we can calculate the spectrum of soft masses. The messenger of SUSY breaking is mostly gravity, however, there are other contributions from gauge and anomaly mediation.

24.3.1 Contributions to the Soft Terms

At tree level, the general soft terms for gravity mediation are given in [520–524]. The models described in this paper contain an additional contribution from the F-term of a scalar field ϕ_2. Following [520, 521, 524], we define

$$F^I \equiv e^{\mathscr{K}/2}\mathscr{K}^{I\bar{J}}\left(\bar{\mathscr{W}}_{\bar{J}} + \bar{\mathscr{W}}\mathscr{K}_{\bar{J}}\right). \tag{24.48}$$

Assuming zero cosmological constant we have the following SUSY breaking soft terms.

SUGRA Effects

Gaugino Masses

The tree level gaugino masses are given by

$$M_a^{(0)} = \frac{g_a^2}{2}F^n\partial_n f_a(S, T_i). \tag{24.49}$$

At tree level, the gauge kinetic function in heterotic string theory is linear in the dilaton superfield S, and only dependent on the T moduli at one loop [Eq. (24.26)]. Note, if $F^{T^i} \gg F^S$ then gaugino masses are one loop suppressed.

A Terms

At tree level, the A terms are given by

$$A_{IJK}^{(0)} = F^n\partial_n\mathscr{K} + F^n\partial_n \log\frac{Y_{IJK}}{\kappa_I\kappa_J\kappa_K}, \tag{24.50}$$

where

$$Y_{IJK} \equiv \frac{\partial^3\mathscr{W}}{\partial\Phi^I\partial\Phi^J\partial\Phi^K} \tag{24.51}$$

and $\mathscr{K}$ is the Kähler potential. Neglecting U dependence, we have

$$\mathscr{K} \supset \Phi_I \, \bar{\Phi}^I \, \kappa_I \quad \text{with} \quad \kappa_I \equiv \prod_i \left(T_i + \bar{T}_i\right)^{-n_I^i}. \tag{24.52}$$

The κ_I are the Kähler metrics for the chiral multiplets, Φ_I, where the A terms are expressed in terms of canonically normalized fields. For matter fields we have[19]

$$A^{(0)}_{IJK} = -\frac{F^S}{(S+\bar{S})} + \sum_i \frac{F^{T_i}}{(T_i+\bar{T}_i)} \left(-1 + n_I^i + n_J^i + n_K^i + (T_i+\bar{T}_i)\frac{\partial_{T_i} Y_{IJK}}{Y_{IJK}}\right). \tag{24.53}$$

In general, there are also tree level contributions to A terms proportional to

$$-\frac{F_{\phi_2}}{\langle\phi_2\rangle}\frac{\partial \log Y_{IJK}}{\partial \log \phi_2}. \tag{24.54}$$

These terms may be dominant, but unfortunately they are highly model dependent. They may give a significant contribution to A_b and A_τ, but in fact we find that the details of the low energy spectrum are not significantly effected.

Scalar Masses

The tree level scalar masses are given by

$$\left(m_I^{(0)}\right)^2 = m^2_{3/2} - F^n \bar{F}^{\bar{m}} \partial_n \partial_{\bar{m}} \log \kappa_I + g_G^2 f \, q_A^I \, \langle D_A \rangle \, \kappa_I, \tag{24.55}$$

where $g_G^2 = 1/ReS_0$ and we have implicitly assumed that the Kähler metric is diagonal in the matter fields. The factor f re-scales the $U(1)_A$ charges q_A from the mini-landscape "benchmark" model 1 [406], so they are consistent with the charges q'_A in our mini-version of the mini-landscape model. We have $q'_A = q_A f = q_A \frac{48\pi^2}{TrQ}\delta_{GS}$ with $\delta_{GS} = \frac{N_f}{4\pi^2}$ [Eq. (24.7)] and $TrQ = \frac{296}{3}$ ([406, Equation E.5]) such that $\frac{Tr(q')}{4\pi^2} = \delta_{GS}$.

Again neglecting U dependence, the Kähler metric for the matter fields depends only on the T moduli, and we find

$$\left(m_I^{(0)}\right)^2 = m^2_{3/2} - \sum_i \frac{n_I^i \left|F^{T_i}\right|^2}{\left(T_i + \bar{T}_i\right)^2} + g_G^2 f \, q_A^I \, \langle D_A \rangle / (2ReT_0)^{n_I^3}. \tag{24.56}$$

[19] Note, with just dilaton and moduli SUSY breaking we can define $\frac{F^S}{(S+\bar{S})} = -\sqrt{3} m_{3/2} \sin(\theta) e^{-i\phi_S}$, $\frac{F^{T_i}}{(T_i+\bar{T}_i)} = -\sqrt{3} m_{3/2} \cos(\theta) e^{-i\phi_i} \Theta_i$ with $\sum_{i=1}^3 \Theta_i^2 = 1$. Then $A^{(0)}_{IJK}$ is independent of the moduli VEVs, only depending on the mixing angles, θ, ϕ_S, Θ_i, ϕ_i.

μ and $B\mu$ Terms

The μ term can come from two different sources:

$$\mathscr{K} \supset Z(T_i + \bar{T}_i, U_j + \bar{U}_j, \ldots)\mathbf{H}^u\mathbf{H}^d, \quad \mathscr{W} \supset \tilde{\mu}(\mathbf{s}_I, T_i, U_j, \ldots)\mathbf{H}^u\mathbf{H}^d. \tag{24.57}$$

In the orbifold models, Kähler corrections have not been computed, so the function Z is *a priori* unknown. Such a term could contribute to the Giudice-Masiero mechanism [117]. When both $\tilde{\mu}$ and Z vanish, the SUGRA contribution to the $\mu/B\mu$ terms vanish. On the other hand, in the class of models which we consider, we know that vacuum configurations exist such that $\tilde{\mu} = 0$ to a very high order in singlet fields. Moreover $\tilde{\mu} \propto \langle\mathscr{W}\rangle$ which vanishes in the supersymmetric limit, but obtains a value w_0 at higher order in powers of chiral singlets. If μ is generated in this way, there is also likely to be a Peccei-Quinn axion [516–519]. Finally, supergravity effects will also generate a $B\mu$ term. In addition, if the theory includes a $\mathbb{Z}_4^R$ symmetry, then $\tilde{\mu} = 0$ to all orders in perturbation theory and can only be generated by non-perturbative effects of order $\langle\mathscr{W}\rangle$.

Loop Corrections

Finally, one can consider loop corrections to the tree level expressions in [520, 521, 524]. This was done in [525, 526], where the complete structure of the soft terms (at one loop) for a generic (heterotic) string model were computed in the effective supergravity limit. Applying the results of [525, 526] to our models and we find, at most, around a 10% correction to the tree level results of [520, 521, 524].[20]

Gauge Mediation

The "mini-landscape" models generically contain vector-like exotics in the spectrum. Moreover it was shown that such states were necessary for gauge coupling unification [439]. The vector-like exotics obtain mass in the supersymmetric limit by coupling to scalar moduli, thus they may couple to the SUSY breaking field ϕ_2. For example, consider the following light exotics to have couplings linear in the field ϕ_2:

$$n_3 \times (\mathbf{3}, 1)_{1/3} + n_2 \times (1, \mathbf{2})_0 + n_1 \times (1, 1)_{-1} + \text{h.c.} \tag{24.58}$$

[20] In estimating this result, we have assumed that the mass terms of the Pauli-Villars fields do not depend on the SUSY breaking singlet field ϕ_2, and that the modular weights of the Pauli-Villars fields obey specific properties.

where the constants n_i denote the multiplicity of states and (see Table 7 of [439])

$$n_3 \leq 4 \text{ and } n_2 \leq 3 \text{ and } n_1 \leq 7. \tag{24.59}$$

The gauge mediated contributions split the gaugino masses by an amount proportional to the gauge coupling:

$$M_3^{(1)}|_{\text{gmsb}} = n_3 \frac{g_3^2}{16\pi^2} \frac{F^{\phi_2}}{\langle\phi_2\rangle}, \tag{24.60}$$

$$M_2^{(1)}|_{\text{gmsb}} = n_2 \frac{g_2^2}{16\pi^2} \frac{F^{\phi_2}}{\langle\phi_2\rangle}, \tag{24.61}$$

$$M_1^{(1)}|_{\text{gmsb}} = \frac{n_3 + 3n_1}{10} \frac{g_1^2}{16\pi^2} \frac{F^{\phi_2}}{\langle\phi_2\rangle}. \tag{24.62}$$

It is interesting to note that this becomes more important as $\langle\phi_2\rangle$ decreases, F^{ϕ_2} increases, or if there are a large number of exotics present.

The scalar masses in gauge mediation come in at two loops, and receive corrections proportional to

$$(m_I)^2\,|_{\text{gmsb}} \sim \left(\frac{1}{16\pi^2}\right)^2 \left(\frac{F_{\phi_2}}{\phi_2}\right)^2. \tag{24.63}$$

Unlike in the case of the gaugino masses, however, the tree level scalar masses are set by the gravitino mass. Typically

$$16\pi^2 m_{3/2} >> \frac{F_{\phi_2}}{\phi_2}, \tag{24.64}$$

and the gauge mediation contribution gives about a 10% correction to the scalar masses, in our case. We will neglect their contributions in the calculation of the soft masses below.

24.3.2 Hierarchy of F-Terms

Note, in Sect. 24.2, we find (roughly)

$$F_T >> F_S \gtrsim F_{\phi_2}, \tag{24.65}$$

for Cases 1, 2 and 3; and

$$F_T \gtrsim F_{\phi_2} >> F_S, \tag{24.66}$$

Table 24.4 The hierarchy of F terms in the five examples of the single condensate model we studied

	Case 1	Case 2	Case 3	Case 4	Case 5
F^S	6.6×10^{-16}	3.7×10^{-16}	4.2×10^{-16}	2.7×10^{-16}	2.1×10^{-16}
F^T	-2.2×10^{-15}	-1.2×10^{-15}	-1.4×10^{-15}	1.6×10^{-15}	2.2×10^{-15}
F^{ϕ_2}	-1.1×10^{-17}	-6.5×10^{-18}	-7.7×10^{-18}	1.9×10^{-16}	1.8×10^{-16}

Note that F^{Φ} is defined in Eq. (24.48). All of the F terms contribute to the soft masses, as they are all within an order of magnitude

for Cases 4 and 5, where

$$F_I \equiv \mathscr{W}_I + \mathscr{W}\mathscr{K}_I. \tag{24.67}$$

When one includes the relevant factors of the Kähler metric, we have (Table 24.4)

$$F^T > F^S >> F^{\phi_2} \tag{24.68}$$

for Cases 1, 2 and 3; and

$$F^T >> F^S \sim F^{\phi_2} \tag{24.69}$$

for Cases 4 and 5. F^S is enhanced by a factor of $\mathscr{K}^{S\bar{S}} \sim (2+2)^2$, while F^{ϕ_2} is decreased by a factor of $\mathscr{K}^{\phi_2\bar{\phi}_2} \sim (2)^{-1/2}$.[21] This means that although the singlet field ϕ_2 was a dominant source of SUSY breaking, it is the least important when computing the soft terms, given the one condensate hidden sector of the simplified model studied in Sect. 24.2.[22]

Natural SUSY Breaking

In Chap. 12 we discussed the little hierarchy problem and a possible resolution in terms of special boundary conditions at the GUT scale. In particular, given the boundary conditions the $SO(10)$ boundary conditions with

$$m_{10} = \sqrt{2}m_{16}, \quad A_0 = -2m_{16} \tag{24.70}$$

there was minimal fine-tuning. In this section we discuss a possible mechanism for generating such boundary conditions in a string scenario.

[21]This is due to the assumed modular weight of the field ϕ_2.

[22]In racetrack models F_S is suppressed by more than an order of magnitude. In these cases F_{ϕ_2} is dominant [466].

Consider Heterotic orbifold models with dilaton/moduli SUSY breaking as discussed in [524]. With just dilaton and moduli SUSY breaking we have

$$\frac{F^S}{(S+\bar{S})} = -\sqrt{3} m_{3/2} \sin(\theta) e^{-i\phi_S}, \quad \frac{F^{T_i}}{(T_i+\bar{T}_i)} = -\sqrt{3} m_{3/2} \cos(\theta) e^{-i\phi_i} \Theta_i. \tag{24.71}$$

θ determines the amount of SUSY breaking of the dilaton sector versus the moduli sector and Θ_i gives the probability for the SUSY breaking contribution of each modulus with $\sum_{i=1}^{3} \Theta_i^2 = 1$.[23] We have the scalar masses for sparticle I, the trilinear couplings and the gaugino masses are given by

$$\begin{aligned} M_a &= \sqrt{3} m_{3/2} \Bigg[\sin\theta e^{-i\phi_s} \frac{k_a ReS}{Ref_a} \\ &\qquad + \cos\theta e^{-i\phi_i} \Theta_i \left(\frac{\left(\alpha_a^i - k_a \delta_\sigma^i\right)(T_i+\bar{T}_i)\hat{G}_2(T_i,\bar{T}_i)}{32\pi^3 Ref_a} \right) \Bigg], \\ m_I^2 &= m_{3/2}^2 (1 - 3\cos^2\theta \sum_{i=1}^{3} n_I^i\, \Theta_i^2), \qquad (24.72) \\ A_{IJK} &= -\sqrt{3} m_{3/2} \Bigg(\sin\theta e^{-i\phi_s} + \cos\theta \\ &\qquad (\sum_{i=1}^{3} e^{-i\phi_i} \Theta_i [-1 + n_I^i + n_J^i + n_K^i + (T_i+\bar{T}_i)\partial_i \log Y_{IJK}]) \Bigg), \end{aligned}$$

where

$$\hat{G}_2(T,\bar{T}) = G_2(T) - \frac{2\pi}{(T+\bar{T})} \quad \text{and} \quad G_2(T) = -4\pi \frac{\partial \log \eta(T)}{\partial T}.$$

Note, that for the special case, $Y_{IJK} \sim h_{IJK} \prod_{i=1}^{3} \eta(T_i)^{\gamma_{T_i}}$ as in Eq. (24.26), we have

$$\begin{aligned} A_{IJK} &= -\sqrt{3} m_{3/2} \Bigg(\sin\theta e^{-i\phi_s} \cos\theta \\ &\qquad + \left[\sum_{i=1}^{3} e^{-i\phi_i} \Theta_i (-1 + n_I^i + n_J^i + n_K^i)(1 - \frac{\pi}{6}(T_i+\bar{T}_i) \right] \Bigg), \end{aligned} \tag{24.73}$$

where we used $\eta(T) \approx e^{-\frac{\pi T}{12}}$ for $T \geq 1$.

[23] I am assuming that only Kähler moduli, T_i, $i = 1,2,3$, contribute to SUSY breaking, i.e. the complex structure moduli, U_i, $i = 1,2,3$, have vanishing F terms.

As a particular example, consider a $\mathbb{Z}_2 \otimes \mathbb{Z}_6'$ orbifold with twist vectors given by $(1/2, 1/2, 0)$, $(1/6, 2/3, 1/6)$ in the three two torii [446]. There are three Kahler moduli in this example. Then we have

$$m_I^2 = m_{3/2}^2(1 - 3(n_I^{T_1}\Theta_1^2 + n_I^{T_3}\Theta_3^2)) \tag{24.74}$$

where we assumed that $\Theta_2 = 0$.

We now assume that the Higgs 10-plet comes from the bulk on the second two torus. Thus $n_{10}^{T_1} = n_{10}^{T_3} = 0,\ n_{10}^{T_2} = 1$. Hence,

$$m_{10} = m_{3/2} \tag{24.75}$$

and therefore,

$$m_{16}^2 = m_{10}^2(1 - 3(n_{16}^{T_1}\Theta_1^2 + n_{16}^{T_3}\Theta_3^2)). \tag{24.76}$$

If the 16-plet lives in the fifth twisted sector with modular weights, $n_{16}^{T_1} = n_{16}^{T_3} = 1/6,\ n_{16}^{T_2} = 2/3$, we have $m_{16}^2 = \frac{1}{2}m_{10}^2$ or

$$m_{10} = \sqrt{2}m_{16}. \tag{24.77}$$

This is our first constraint, that $m_{10} \equiv (m_{H_u} + m_{H_d})/2 \approx \sqrt{2}m_{16}$.

Assuming $\sin\theta \sim 0$, we have

$$A_{(16\ 16\ 10)} \sim -2\sqrt{\frac{2}{3}}m_{16}\left[e^{-i\phi_1}\Theta_1(1 - \frac{\pi}{6}(T_1 + \bar{T}_1)) + e^{-i\phi_3}\Theta_3(1 - \frac{\pi}{6}(T_3 + \bar{T}_3))\right].$$

If $T_{1,3} = \bar{T}_{1,3} \sim 2$, then $A_{(16\ 16\ 10)} = A_0 = -2m_{16}$ for $e^{-i\phi_1}\Theta_1 = \frac{1}{\sqrt{2}}e^{-i\gamma}$, $e^{-i\phi_3}\Theta_3 = \frac{1}{\sqrt{2}}e^{+i\gamma}$ and $\gamma = 5\pi/6$. Of course, it would require the dynamics of stabilizing moduli and SUSY breaking to fix these particular values of $T_i,\ \phi_i,\ \Theta_i$.

24.4 Conclusions

As a candidate theory of all fundamental interactions, string theory should admit at least one example of a four-dimensional vacuum which contains particle physics and early universe cosmology consistent with the two standard models. In this context, the "mini-landscape" of heterotic orbifold constructions [405–407, 413, 414] or the heterotic orbifold model with $\mathbb{Z}_4^R$ symmetry [276] provides us with very promising four-dimensional perturbative heterotic string vacua. Their low-energy effective field theory was shown to resemble that of the MSSM, assuming non-zero VEVs for certain blowup moduli fields which parametrize resolutions of the orbifold fixed points along F- and D-flat directions in global supersymmetry.

In this chapter we have dealt with the task of embedding the globally supersymmetric constructions of the heterotic into supergravity and then stabilizing the moduli of these compactifications, including their orbifold fixed point blowup moduli. The blowup moduli appear as chiral superfields contained in the twisted sectors of the orbifolded heterotic string theory. They are singlets under all standard model gauge groups, but are charged under several unwanted $U(1)$ gauge symmetries, including the universal anomalous $U(1)_A$ gauge symmetry of the heterotic string. Note, moduli stabilization of string compactifications is a crucial precondition for comparing to low energy data, as well as for analyzing any early universe cosmology, such as inflation, in a given construction.

Section 24.1 served the purpose of reviewing the ingredients and structure of the heterotic 4D $N = 1$ supergravity inherited from orbifold compactifications of the 10D perturbative $E_8 \otimes E_8$ heterotic string theory. The general structure of these compactifications results in:

1. a standard Kähler potential for the bulk volume and complex structure moduli, as well as the dilaton, together with
2. gaugino condensation in the unbroken sub-group of the hidden E_8, and
3. the fact that the non-perturbative (in the world-sheet instanton sense) Yukawa couplings among the twisted sector singlet fields contain terms explicitly breaking the low-energy $U(1)_R$-symmetry.

We have shown in Sect. 24.2 that these three general ingredients, present in all of the semi-realistic heterotic orbifold constructions, effectively realize a KKLT-like setup for moduli stabilization. Here, the existence of terms explicitly breaking the low-energy $U(1)_R$-symmetry at high order in the twisted sector singlet fields is the source of the effective small term w_0 in the superpotential, which behaves like a constant with respect to the heterotic dilaton [371]. Utilizing this, the presence of just a single condensing gauge group in the hidden sector (in contrast to the racetrack setups in the heterotic literature) suffices to stabilize the bulk volume T (and, by extension, also the bulk complex structure moduli U), as well as the dilaton S at values $\langle \mathrm{Re}\ T\rangle \sim 1.1$–$1.6$ and $\langle \mathrm{Re}\ S\rangle \sim 2$. These are the values suitable for perturbative gauge coupling unification into $SU(5)$- and $SO(10)$-type GUTs distributed among the orbifold fixed points. Note, we have shown this explicitly for the case of one T modulus and a dilaton, however, we believe that all bulk moduli will be stabilized near their self-dual points [509, 513].

At the same time, the near-cancelation of the D-term of the universal anomalous $U(1)_A$-symmetry stabilizes non-zero VEVs for certain gauge invariant combinations of twisted sector singlet fields charged under the $U(1)_A$. This feature in turn drives non-vanishing F-terms for some of the twisted sector singlet fields. Thus, together with the F-terms of the bulk volume moduli inherited from modular invariance, it is sufficient to uplift the AdS vacuum to near-vanishing cosmological constant. The effects from the bulk moduli stabilization and supersymmetry breaking, transmitted through supergravity, generically suffice to stabilize all of the twisted sector singlet fields at non-zero VEVs [453]. On the other hand, in the heterotic construction with $\mathbb{Z}_4^R$ symmetry it was argued that most moduli are, in fact, already stabilized at the string scale. This, by itself, would justify using a simplified model of SUSY breaking with only a few moduli.

The structure of the superpotential discussed in this chapter, $\mathscr{W} \sim w_0 e^{-bT} + \phi_2\, e^{-aS-b_2T}$, behaves like a 'hybrid KKLT' with a single-condensate for the dilaton S, but as a racetrack for the T and, by extension, also for U moduli. An additional matter F_{ϕ_2} term driven by the cancelation of the anomalous $U(1)_A$ D-term seeds successful up-lifting.

We note the fact that the effective constant term in the superpotential, w_0, does not arise from a flux superpotential akin to the type IIB case. This leaves open (for the time being) the question of how to eventually fine-tune the vacuum energy to the 10^{-120}-cancelation necessary.

To conclude, we have given a mechanism for moduli stabilization and supersymmetry breaking for the perturbative heterotic orbifold compactifications. It relies on the same variety and number of effective ingredients as the KKLT construction of type IIB flux vacua and thus represents a significant reduction in necessary complexity, compared to the multi-condensate racetrack setups utilized so far. When applied to a simplified analog of the heterotic orbifold compactifications, which give the MSSM at low energies, it leads to fully stabilized 4D heterotic vacua with broken supersymmetry and a small positive cosmological constant. Moreover, most of the low energy spectrum could be visible at the LHC.

24.5 The Role of Holomorphic Monomials

Supersymmetry can be broken by either F terms or D terms. In a generic supersymmetric gauge theory, $D = 0$ is satisfied only along special directions in moduli space. These directions are described by holomorphic, gauge invariant monomials (HIMs) [448, 527, 528]. The moduli space of a general heterotic string model is significantly more complex than that of our simple models. Not only are there many more fields in the picture, there are also many more gauge groups.

Consider a theory with gauge symmetry $U(1)^\rho \otimes U(1)_A$, where A stands for anomalous. The $D = 0$ constraints are

$$D_{a \neq A} \sim \sum_i q_i^a\, |\phi_i|^2 = 0. \tag{24.78}$$

A generic HIM can be written in terms of fields ϕ_i with charges q_i^j

$$\mathscr{H}\,[\phi_i] = \prod_i \phi_i^{n_i}, \quad n_i > 0, \tag{24.79}$$

such that

$$\sum_i n_i q_i^j = 0, \quad \forall j \neq A. \tag{24.80}$$

The requirement that $n_i > 0$ is a reflection of the holomorphicity of $\mathscr{H}$, while the requirement that the sum over n_i (weighted by the charges) vanishes is a reflection of the gauge invariance. The general HIM in Eq. (24.79) relates the VEVs of the fields ϕ as follows:

$$\frac{|\phi_1|}{\sqrt{n_1}} = \frac{|\phi_2|}{\sqrt{n_2}} = \cdots . \tag{24.81}$$

Given this relationship, one can show that Eq. (24.78) can be satisfied. Notice that no scale is introduced in Eq. (24.81): the HIMs (in general) only constrain the relative magnitudes of the ϕ VEVs, and gives no information about their phases or their absolute magnitudes.

The procedure for dealing with an anomalous $U(1)_A$ works the same way. Instead of Eq. (24.78), one has

$$D_A \sim \sum_i q_i^A |\phi_i|^2 + \xi = 0, \tag{24.82}$$

and we will assume that $\xi > 0$. In this case, one needs to find a monomial which is holomorphic and gauge invariant under all of the ρ U(1) factors, but which carries a net negative charge under the anomalous $U(1)_A$ [448, 527, 528]. The situation is different than the case with non-anomalous symmetries, as a mass scale is introduced into the problem.

In a heterotic string orbifold, the FI term is generated by the mixed gauge-gravitational anomaly, and is canceled by the Green-Schwarz mechanism, which forces singlets to get VEVs of order the FI scale (typically $\sim M_S$). Usually, several singlets participate in this cancellation, all receiving VEVs of the same order. In the "mini-landscape" models [406], supersymmetric vacua were obtained, prior to the consideration of any non-perturbative effects. A holomorphic gauge invariant monomial was found which is invariant under all other $U(1)$s but with net charge under $U(1)_A$ opposite to that of the FI term. This composite field necessarily gets a non-zero VEV to cancel the FI term. Our field ϕ_2 in the simple model gives mass to the vector-like exotics of the hidden sector and thus it also appears in the non-perturbative superpotential. In a more general heterotic model, ϕ_2 would be replaced by an HIM which also cancels the FI term.

Chapter 25
Other String Constructions

25.1 Smooth Heterotic String Constructions

In the previous chapters we have shown that heterotic orbifold models can provide the UV completion of orbifold GUT field theories. They provide a "fertile patch" in the string landscape. They naturally contain local GUTs where complete families of quarks and leptons reside. This leads to non-trivial Yukawa sectors with light active neutrinos due to a See-Saw mechanism. The models are consistent with gauge coupling unification and suppression of proton decay rates. Hundreds of MSSM-like models have been obtained. In this chapter we focus on finding MSSM-like models in smooth heterotic string constructions. This requires compactifying the 6 extra dimensions on a Calabi-Yau three-fold. Many models starting with an $E_8 \times E_8$ gauge symmetry have been constructed which have the gauge structure and particle content of the MSSM. In most, if not all, cases this has been accomplished by breaking one of the E_8 groups to either $SU(5)$ or $SO(10)$ with gauge flux and then breaking further to the SM gauge group via a Wilson line (for example, see [529, 530]). Hundreds of MSSM-like models have been found by using Abelian fluxes to break one E_8 to $SU(5)$ and then a Wilson line to break further to the SM gauge group [531–533]. In addition non-trivial Yukawa matrices are also obtained given by effective higher dimensional operators proportional to products of SM singlet moduli [532]. Once again it is clear that to find the MSSM in the string landscape one is greatly assisted by first building a supersymmetric GUT in the landscape.

25.2 Type II String Constructions

Type II string constructions require a completely different paradigm. Non-abelian gauge groups are obtained by embedding stacks of D-branes in the Calabi-Yau three-fold. In the cases most studied the Calabi-Yau three-fold is taken to be an orbifold

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with the SM obtained by taking a stack of 3 D-branes, intersecting a stack of 2 D-branes and finally intersecting several single D-branes. This leads to the gauge group $SU(3)\times SU(2)\times U(1)^n$. Quarks and leptons are then found at the intersections of the D-branes (for a review, see [534, 535]). Supersymmetric GUT models have also been constructed, but they suffer from several severe problems. The extra $U(1)$ symmetries prevent a renormalizable Yukawa coupling for the top quark. They also forbid Majorana masses for right-handed neutrinos. However it has been shown that non-perturbative instanton effects can still generate these forbidden terms, albeit somewhat suppressed. A solution around this problem has been found in the context of F-theory, which is in fact a non-perturbative limit of type II string models.

25.3 F-Theory

$SU(5)$ GUT F-theory models have been constructed with three families of quarks and leptons and one pair of Higgs $\mathbf{5}$ and $\bar{\mathbf{5}}$s. These are either defined locally near the GUT brane, [536–540] or globally on a Calabi-Yau four-fold (see for example, [541]). In all cases the $SU(5)$ GUT symmetry is broken to the SM gauge group and Higgs doublet-triplet splitting is accomplished with non-trivial hypercharge flux. This typically leads to large threshold corrections to gauge coupling unification [539, 542]. In addition, R parity, necessary for forbidding dimension 4 baryon and lepton number violating operators, requires an additional $U(1)_{B-L}$ symmetry [543]. This symmetry should however be spontaneously broken above the weak scale. Finally, dimension 5 baryon and lepton number violating operators are typically only suppressed by the string scale, which may not be sufficient.

The first problem of large threshold corrections to gauge coupling unification can be addressed using Wilson line breaking of the GUT symmetry. The only $SU(5)$ F-theory model with this solution is given in [544]. This model, however, suffers from having massless vector-like exotics and four pairs of massless Higgs doublets. A resolution of the problem of massless vector-like exotics is elusive. A solution to the problem of dimension 5 baryon and lepton number violating operators could be resolved with the additional $\mathbb{Z}_4^R$ symmetry.

25.4 M-Theory

M-theory is supergravity in 11 dimensions. When some dimensions are compactified on particular compact manifolds the theory has been shown to contain the spectrum of heterotic, Type I or II string theories [545]. Gauge groups and massless fermions can be found at singularities of M-theory [546, 547]. There are also attempts at obtaining SUSY GUTs with MSSM-like low energy theories from M-theory, for example, see [548]. Such attempts are especially difficult, since to obtain an $N = 1$ SUSY theory in 4D one must compactify the 11 dimensions on a 7D manifold with G_2 holonomy, known as a Joyce manifold [549]. Unfortunately this is an extremely difficult proposition.

Chapter 26
Epilogue

In this book we have discussed an evolution of SUSY GUT model building. We saw that 4D SUSY GUTs have many virtues. However there are some problems which suggest that these models may be difficult to derive from a more fundamental theory, i.e. string theory.[1] We then discussed orbifold GUT field theories which solve two of the most difficult problems of 4D GUTs, i.e. GUT symmetry breaking and Higgs doublet-triplet splitting. Finally, we showed how some orbifold GUTs can find an ultra-violet completion within the context of heterotic string theory.

The flood gates are now wide open. In a "mini-landscape" analysis [406] we obtained many models with features like the MSSM: SM gauge group with three families and vector-like exotics which can, in principle, obtain large mass. The models have an exact R-parity and non-trivial Yukawa matrices for quarks and leptons. In addition, neutrinos obtain mass via the See-Saw mechanism. We showed that gauge coupling unification can be accommodated [439]. We were also able to obtain an MSSM-like model with a $\mathbb{Z}_4^R$ symmetry which can resolve the μ and proton decay problems of GUTs [276]. Other MSSM-like models have been obtained with the heterotic string compactified on a $T^6/\mathbb{Z}_{12}$ orbifold [438] or using free fermionic constructions [550]. Smooth heterotic and F theory constructions have also been used to obtain MSSM-like models. In all cases, embedding a GUT symmetry explicitly into the construction allows for the success of finding MSSM-like models with the correct gauge groups and matter content.

Of course, this is not the end of the story. It is just the beginning. We must still obtain predictions for the LHC. This requires stabilizing the moduli and breaking supersymmetry. In fact, these two conditions are *not* independent, since once SUSY is broken, the moduli will be stabilized. The size of the extra dimensions, as well as all gauge and Yukawa couplings are manifestly dependent on the values of these moduli. The scary fact is that the moduli have to be stabilized at just the right values

[1] Note, string theories have the possibility of providing a theory of everything since they also include a consistent theory of quantum gravity.

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to be consistent with low energy phenomenology. String cosmology also has an immense body of literature (see for example the book by Baumann and McAllister [245].)

Supersymmetric grand unification is beautiful, but ultimately we want to know if it is a piece of reality. In order to know this, we must necessarily see the new supersymmetric particles at the LHC and/or some evidence for proton decay. We have been waiting for a long time for this discovery, but unfortunately not all scientific discoveries occur on the timescale of a lifetime.

Chapter 27
Problems

27.1 Problem 1

1. Given the Lagrangian for the Standard Model, the Z boson couples to quarks and leptons in the Dirac basis proportional to the vector and axial vector coupling constants, g_V and g_A [Eq. (2.17)]. Evaluate $g_V, \; g_A$ for up quarks and down quarks.
2. Write the first term in Eq. (2.20) using Dirac notation.

27.2 Problem 2

1. Consider a chiral superfield Φ which is a gauge singlet and has component fields given by

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}(\theta \; \psi(y)) + (\theta \; \theta) F_\phi(y) \tag{27.1}$$

   It has a Kähler potential

$$\mathscr{K} = \Phi^* \; \Phi \tag{27.2}$$

   and superpotential

$$\mathscr{W} = \frac{1}{3} \lambda \; \Phi^3 + \frac{1}{2} m \; \Phi^2 . \tag{27.3}$$

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Calculate the one loop correction to the ϕ mass assuming soft SUSY breaking terms

$$-\mathscr{L}_{SUSYbreaking} = m_\phi^2\, \phi^*\, \phi + \frac{1}{3}\, A\, \lambda\, \phi^3 + \frac{1}{2}\, B\, m\, \phi^2 + h.c. \tag{27.4}$$

i.e. just calculate the quadratic correction to the scalar masses. Note,

$$\phi = \frac{1}{\sqrt{2}}\,(\phi_1 + i\phi_2) \tag{27.5}$$

where $\phi_1,\ \phi_2$ are real scalar fields. All parameters are real.

27.3 Problem 3

This problem is on the O'Raifeartaigh mechanism for spontaneous SUSY breaking
Given the SUSY Lagrangian with chiral superfields $\{A,\ B,\ C\}$, Kähler potential

$$\mathscr{K} = A^\dagger\, A + B^\dagger\, B + C^\dagger\, C \tag{27.6}$$

and superpotential

$$\mathscr{W} = A(B^2 + M^2) + \mu\, C\, B. \tag{27.7}$$

The mass parameters M and μ are real and take $M^2 > \frac{\mu^2}{2}$. Find the ground state of the theory for arbitrary values of $\langle C\rangle = c_0$. Define the fields $G,\ L$ via the unitary transformation

$$\begin{pmatrix} G \\ L \end{pmatrix} = N^{-1} \begin{pmatrix} \langle\frac{\partial\mathscr{W}}{\partial A}\rangle|_0 & \langle\frac{\partial\mathscr{W}}{\partial C}\rangle|_0 \\ -\langle\frac{\partial\mathscr{W}}{\partial C}\rangle|_0^* & \langle\frac{\partial\mathscr{W}}{\partial A}\rangle|_0^* \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} \tag{27.8}$$

where $N = \left(|\langle\frac{\partial\mathscr{W}}{\partial A}\rangle|_0^2 + |\langle\frac{\partial\mathscr{W}}{\partial C}\rangle|_0^2\right)^{1/2}$. Re-write the Lagrangian in terms of the fields $G,\ L$. Calculate the spectrum of states by expanding around the vacuum solutions with the definitions, $\langle A\rangle = a_0$, $\langle B\rangle = b_0$ and $\langle C\rangle = c_0$. Calculate the supertrace of the mass squared operator, i.e.

$$Str\mathscr{M}^2 = \sum_J (-1)^{2J}(2J+1)Tr\mathscr{M}_J^2. \tag{27.9}$$

What is the mass of the field G?

27.4 Problem 4

1. In the lecture, it was argued that a complete Standard Model generation can be fit in to the $\bar{\mathbf{5}} + \mathbf{10}(+1)$ representations of SU(5). In general, an element of the group SU(5) can be written as

$$U = e^{i\epsilon^A T_A} = \mathbb{1} + i\epsilon^A T_A + \mathscr{O}(\epsilon^2). \tag{27.10}$$

The fundamental representation of SU(5) transforms as follows:

$$\mathbf{5}'^{\alpha} = U^{\alpha}{}_{\beta}\mathbf{5}^{\beta} = \left\{\delta^{\alpha}{}_{\beta} + i\epsilon^A (T_A)^{\alpha}{}_{\beta} + \mathscr{O}(\epsilon^2)\right\}\mathbf{5}^{\beta}. \tag{27.11}$$

(a) Using this, show how the two index anti-symmetric tensor ($\mathbf{10}^{\alpha\beta}$) transforms under infinitesimal gauge transformations.

(b) Given the transformation of the conjugate representation, $\bar{\mathbf{5}}$,

$$\bar{\mathbf{5}}'_{\alpha} = U^{*\alpha}_{\ \ \beta}\, \bar{\mathbf{5}}_{\beta}. \tag{27.12}$$

Define the transformation of the adjoint representation $A^{\alpha}{}_{\beta}$ via

$$A'^{\alpha}{}_{\beta} = U^{\alpha}{}_{\gamma}\, U^{*\beta}{}_{\delta}\, A^{\gamma}{}_{\delta}. \tag{27.13}$$

Show how $A^{\alpha}{}_{\beta}$ transforms under infinitesimal gauge transformations.

2. In order to break SU(5) to the MSSM, we need to introduce some scalar GUT Higgs multiplets, which must transform in the adjoint (**24**) representation, and a scalar potential. (Why won't smaller representations work for this job?) Given that the Higgs scalar Σ has the following covariant derivative:

$$D_{\mu}\Sigma = \partial_{\mu}\Sigma + ig\left[V_{\mu}^{A}T_A, \Sigma\right], \tag{27.14}$$

show that the **X** gauge bosons obtain mass

$$m_{\mathbf{X}}^2 = \frac{25}{4}g^2V^2. \tag{27.15}$$

Hint: Take $\langle\Sigma\rangle = V\ \ \mathrm{diag}(1, 1, 1, -3/2, -3/2)$, where $V \sim M_G$.

27.5 Problem 5

Given the $SU(5)$ GUT breaking and Higgs doublet-triplet splitting sectors in the notes

$$\mathscr{W} = \frac{\lambda}{3}\, Tr(\Sigma)^3 - \frac{M}{2}\, Tr(\Sigma)^2 + \bar{H}\,(\lambda'\ \Sigma\ - M')\, H, \tag{27.16}$$

calculate the spectrum of massive states (assuming $\langle \Sigma \rangle$ breaks $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)_Y$) and show that only the states in the MSSM are present in the low energy theory below the GUT scale of order M/λ. Note, first show that the symmetry breaking VEV preserves supersymmetry. Therefore the massive spectrum must form a massive N = 1 supermultiplet. For the gauge sector the massive gauge super multiplet includes a massive vector boson, a real scalar and a Dirac fermion (all transforming in the same representation under the unbroken gauge symmetry). A massive chiral super multiplet includes two complex scalar fields and a Dirac fermion (again, all transforming in the same representation under the unbroken gauge symmetry).

27.6 Problem 6

Given the one loop RG running of the three gauge couplings, $\alpha_i, \ i = 1, 2, 3$, in the MSSM and the low energy boundary conditions considered in the text, find α_G and M_G for $N_{fam} = 4, \ \ N_{(H_u+H_d)} = 1$ and for $N_{fam} = 3, \ \ N_{(H_u+H_d)} = 2$. Are there any problems with either of these choices?

27.7 Problem 7

In Fig. 27.1 we have the multiplication table for the quaternionic group. The group has 8 elements and 5 irreducible representations, i.e. 1 doublet and 4 singlets.

	e	a_1	a_2	a_3	a_4	a_5	a_6	a_7
e	e	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_1	a_1	e	a_3	a_2	a_5	a_4	a_7	a_6
a_2	a_2	a_3	a_1	e	a_6	a_7	a_5	a_4
a_3	a_3	a_2	e	a_1	a_7	a_6	a_4	a_5
a_4	a_4	a_5	a_7	a_6	a_1	e	a_2	a_3
a_5	a_5	a_4	a_6	a_7	e	a_1	a_3	a_2
a_6	a_6	a_7	a_4	a_5	a_3	a_2	a_1	e
a_7	a_7	a_6	a_5	a_4	a_2	a_3	e	a_1

Fig. 27.1 The multiplication table for the quaternionic group

1. Show that the matrices in Eq. (27.17) satisfy the quaternionic multiplication rules.

$$D_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; D_{a_1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix};$$
$$D_{a_2} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; D_{a_3} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix};$$
$$D_{a_4} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; D_{a_5} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$
$$D_{a_6} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; D_{a_7} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{27.17}$$

2. Given the two doublets $\overrightarrow{X}$, $\overrightarrow{Y}$ with components defined by

$$\overrightarrow{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{27.18}$$

and

$$\overrightarrow{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{27.19}$$

show that the inner product

$$\overrightarrow{X} \cdot \overrightarrow{Y} = x_1\, y_2 - x_2\, y_1 \tag{27.20}$$

is invariant under quaternionic transformations.

27.8 Problem 8

$U(1)$ D terms due to $U(1)$ gauge symmetries broken at a GUT scale have been used in the literature to provide soft SUSY breaking mass terms at the weak scale. Such D terms have been used to split sparticle masses, for example, the $U(1)_X$ D term in $SO(10)$ models when $SO(10)$ is broken to $SU(5) \times U(1)_X$ was used to split the H_u and H_d soft masses. In addition, $U(1)$ D terms coming from non-Abelian flavor symmetries are argued to be a problem for flavor physics even when the $U(1)$ was spontaneously broken at a scale much above the weak scale. For this reason it has been suggested that non-Abelian discrete symmetries could be used to avoid this D term problem. In this problem we show that if the gauge symmetry is spontaneously broken at a scale, M, which is much larger than the soft SUSY breaking scale, m, then the $U(1)$ quartic term decouples from low energy physics. However, the $U(1)$ D terms remains in the effective low energy theory.

You are given a supersymmetric $U(1)$ gauge theory with heavy charged chiral superfields $\phi_+,\ \phi_-$ with charge ± 1 and gauge coupling, $g \sim 1$. In addition there are also massless chiral superfields, $\phi_i,\ i = 1, \cdots, N$, with $U(1)$ charge, q_i. You also are given a singlet chiral superfield, X. The superpotential for the theory is $\mathscr{W} = X(\phi_+\,\phi_- - M^2)$. We thus have a scalar potential given by

$$\begin{aligned} V = &|\phi_+\,\phi_- - M^2|^2 + |X|^2(|\phi_+|^2 + |\phi_-|^2) \\ &+ \frac{g^2}{2}(|\phi_+|^2 - |\phi_-|^2 + \sum_i q_i|\phi_i|^2)^2 + m_+^2|\phi_+|^2 + m_-^2|\phi_-|^2 \end{aligned} \tag{27.21}$$

where the mass parameter, $M \gg m_+,\ m_-$, and $m_\pm$ are soft SUSY breaking masses. We take $m^2 = \frac{m_+^2 + m_-^2}{2}$ and $\delta = \frac{m_+^2 - m_-^2}{m^2} \sim 1$.

Calculate the $U(1)$ quartic term and D-term in the low energy theory for energies much below M.

27.9 Problem 9

Given the Lagrangian for a real scalar field, ϕ,

$$\mathscr{L} = \frac{1}{2} g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - V(\phi) \tag{27.22}$$

where $g_{\mu\nu}$ is the metric with signature $(+ - --)$ and $V(\phi)$ is the scalar potential. The stress-energy tensor for the scalar field, $T_{\mu\nu}$, is given by

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\,\mathscr{L}. \tag{27.23}$$

We identify the energy density and pressure of the field by

$$T_{00} = \rho_\phi, \quad T_{ii} = p_\phi. \tag{27.24}$$

Assume

$$V(\phi) = \frac{1}{2}\,m^2\,\phi^2 \tag{27.25}$$

and, in the early universe, the field can be described by a coherent state with central value $\phi(t)$, independent of the spatial coordinate, $\overrightarrow{x}$, and initial value, $\phi_i = \phi(0)$. Write down the equations of motion of the field, $\phi(t)$, in the Robertson-Walker background. Assume that the total energy of the universe has two pieces, a radiation component with $\rho_r = \kappa\, T^4$ and ρ_ϕ. Determine the solution of the equations of motion for $\rho_r \gg \rho_{\phi_i}$ and also in the opposite limit, when $\rho_r \ll \rho_{\phi_i}$. The initial conditions for the ϕ field are $\phi(0) = \phi_i = m = 100\,\text{GeV}$ and $\dot{\phi}(0) = 0$. Describe the solution in the two limits in words and plot the solution for $\phi(t)$.

27.10 Problem 10

In this homework assignment :

(a) Prove Eq. (14.36) and then use it to prove Eqs. (14.37)–(14.40).
(b) Then given Eqs. (14.41), (14.42) show that the mode expansions are given by Eqs. (14.43)–(14.46).
(c) Finally, prove Eq. (14.48).

27.11 Problem 11

(a) Prove Eq. (14.79) in the text.
(b) Following [301], prove Eq. (14.105).

27.12 Problem 12

Evaluate the one-loop renormalization of the gauge coupling constants in the orbifold GUT theory with the sum over all KK modes.

27.13 Problem 13

Consider the $\mathbb{Z}_6$-II orbifold compactification of the heterotic $E_8 \times E_8$ string with the twist vector

$$\mathbf{v}_6 = \frac{1}{6}(1, 2, -3, 0) \tag{27.26}$$

acting on an $\text{G}_2 \times \text{SU}(3) \times \text{SO}(4)$ torus. Consider the gauge shift V, which allows for a local SO(10), given by

$$V_6 = \frac{1}{6}\,(22200000)\,(11000000)\,. \tag{27.27}$$

Show that with this gauge shift the left-moving momenta P which satisfy

$$P \cdot V_6 \;=\; 0 \mod 1\,, \quad P^2 \;=\; 2\,, \quad P \in \Gamma^8 \tag{27.28}$$

are roots of $SO(10)$ (up to extra group factors). In fact, this defines the "local" gauge symmetry in the T_1 sector, for states residing at the origin in the G_2 and SU(3) tori and at the two fixed points in the SO(4) torus which are unaffected by the W_2 Wilson line along the e_6 direction (see Fig. 20.6).[1]

Show that the massless states of the first twisted sector are guaranteed to contain **16**-plets of $SO(10)$ at the fixed points with $SO(10)$ symmetry.

[1] We assume that there are only two Wilson lines, W_3 in the SU(3) torus and W_2 in the SO(4) torus.

Chapter 28
Solutions to Problems

28.1 Solution to Problem 1

1. We have

$$g_V^u = \frac{1}{2} - \frac{4}{3} \sin^2 \theta_W, \quad g_A^u = \frac{1}{2} \text{and} \tag{28.1}$$

$$g_V^d = -\frac{1}{2} + \frac{2}{3} \sin^2 \theta_W, \quad g_A^d = -\frac{1}{2}. \tag{28.2}$$

2.

$$-\mathscr{L}_{Yukawa} \supset \epsilon_{\alpha\beta}\ \bar{\psi}_{e_i}\ Y_e^{ij}\ P_L\ \psi_{l_j}^{\alpha}\ H_d^{\beta} + h.c. \tag{28.3}$$

where $\psi_l = (\begin{matrix}\psi_\nu \\ \psi_e\end{matrix})$.

28.2 Solution to Problem 2

The scalar potential is calculated from the superpotential. We have

$$V = (m^2 + m_\phi^2)\ |\phi|^2 + \lambda^2\ |\phi|^4 + \lambda\ m\ (\phi + \phi^*)\ |\phi|^2 + \frac{\lambda}{3}\ A\ (\phi^3 + \phi^{*3}) + \frac{B\ m}{2}\ (\phi^2 + \phi^{*2}). \tag{28.4}$$

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The scalar mass terms are given by the quadratic terms in V. We have

$$V_{mass} \supset \frac{1}{2}\,(m^2 + m_\phi^2 + B\,m)\,\phi_1^2 + \frac{1}{2}\,(m^2 + m_\phi^2 - B\,m)\,\phi_2^2 = \frac{1}{2}\,m_1^2\,\phi_1^2 + \frac{1}{2}\,m_2^2\,\phi_2^2. \tag{28.5}$$

The higher order terms in V determine the scalar self interactions. We have

$$V_{int} = \frac{\lambda^2}{4}\,(\phi_1^2 + \phi_2^2)^2 + \frac{\lambda\,m}{\sqrt{2}}\,\phi_1\,(\phi_1^2 + \phi_2^2) + \frac{\lambda\,A}{3\sqrt{2}}\,(\phi_1^3 - 3\,\phi_1\,\phi_2^2). \tag{28.6}$$

There is also a Yukawa interaction given by

$$\begin{aligned} \mathscr{L}_\psi &= \psi^*\,i\bar{\sigma}^\mu\,\partial_\mu\,\psi - \frac{1}{2}\,[(m + 2\,\lambda\,\phi)(\psi\,\psi) + h.c.] \\ &= \frac{1}{2}\,[\bar{\Psi}\,i\gamma^\mu\,\partial_\mu\,\Psi - m\,\bar{\Psi}\,\Psi] - \frac{\lambda}{\sqrt{2}}\,(\phi_1\,\bar{\Psi}\,\Psi - i\,\phi_2\,\bar{\Psi}\,\gamma_5\,\Psi) \end{aligned} \tag{28.7}$$

using $(\psi\psi) + h.c. = \bar{\Psi}\,\Psi$ and $(\psi\psi) - h.c. = -\bar{\Psi}\,\gamma_5\,\Psi$ with $\Psi = \begin{pmatrix} \psi \\ i\sigma_2\,\psi^* \end{pmatrix} = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}$. In Fig. 28.1 we show the Feynman diagrams contributing to the quadratic divergent corrections to the ϕ_1 scalar mass. The diagrams for ϕ_2 are the same with slightly different vertices. All other Feynman diagrams have two scalars in the loop and are thus only log divergent.

Let's first consider the Feynman propagator for Majorana fermions. The kinetic term in the Lagrangian is given by the first two terms in Eq. (28.7). We have

$$\frac{1}{2}\,(\bar{\psi}_{\dot{\alpha}}\ \psi^\beta)\begin{pmatrix} \bar{\sigma}^{\mu\,\dot{\alpha}\alpha}p_\mu & -m\delta^{\dot{\alpha}}{}_{\dot{\beta}} \\ -m\delta_\beta{}^\alpha & \sigma^\mu{}_{\beta\dot{\beta}}p_\mu \end{pmatrix}\begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\beta}} \end{pmatrix}. \tag{28.8}$$

This gives us the propagators in momentum space in terms of Weyl spinors, $\psi_\alpha,\ \ \bar{\psi}^{\dot{\alpha}}$

$$\langle\psi_\alpha\ \ \bar{\psi}_{\dot{\beta}}\rangle_0 = \tfrac{i\sigma^\mu{}_{\alpha\dot{\beta}}p_\mu}{p^2 - m^2 + i\epsilon} \tag{28.9}$$

$$\langle\psi_\alpha\ \ \psi^\beta\rangle_0 = \tfrac{im\delta_\alpha{}^\beta}{p^2 - m^2 + i\epsilon}. \tag{28.10}$$

In terms of the 4 component spinor, Ψ, we have

$$\langle\Psi\ \Psi^\dagger\gamma_0\rangle_0 = \begin{pmatrix} \langle\psi_\alpha\psi^\beta\rangle_0 & \langle\psi_\alpha\bar{\psi}_{\dot{\beta}}\rangle_0 \\ \langle\bar{\psi}^{\dot{\alpha}}\psi^\beta\rangle_0 & \langle\bar{\psi}^{\dot{\alpha}}\bar{\psi}_{\dot{\beta}}\rangle_0 \end{pmatrix} = \frac{1}{(p^2 - m^2 + i\epsilon)}\begin{pmatrix} im\delta_\alpha{}^\beta & i\sigma^\mu{}_{\alpha\dot{\beta}}p_\mu \\ i\bar{\sigma}^{\mu\dot{\alpha}\beta}p_\mu & im\delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} \tag{28.11}$$

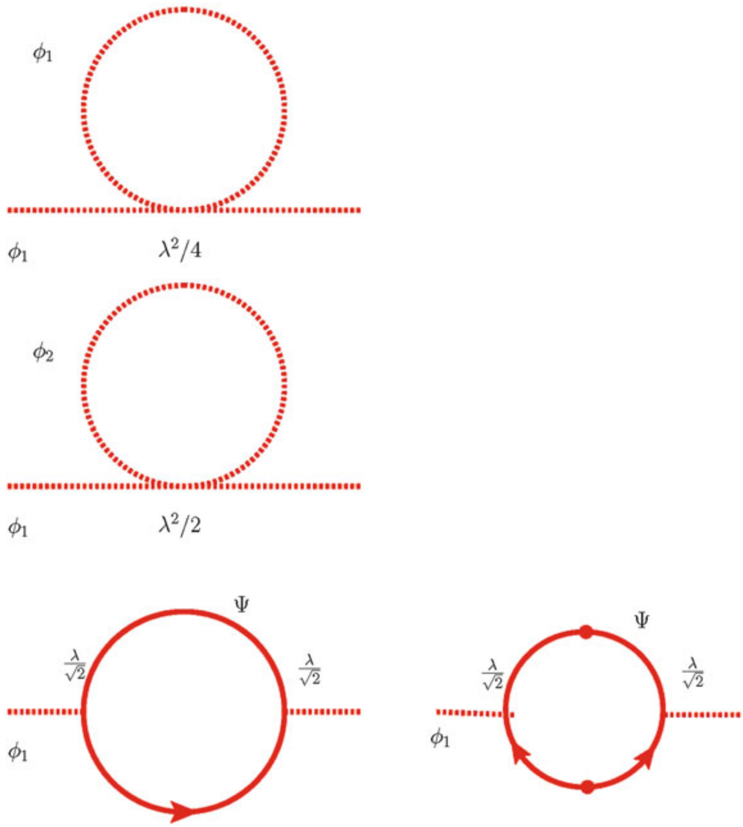

Fig. 28.1 Feynman diagrams contributing to quadratic divergent corrections to the ϕ_1 scalar mass

or

$$\langle \Psi \, \bar{\Psi} \rangle_0 = i \frac{\gamma^\mu p_\mu + m}{(p^2 - m^2 + i\epsilon)}. \tag{28.12}$$

For Majorana 4 component spinors we also have

$$\begin{aligned} \langle \Psi \, \Psi^T \rangle_0 &= \langle \Psi \, \bar{\Psi} \rangle_0 \, C^{-1} = i \tfrac{(\gamma^\mu p_\mu + m) C^{-1}}{(p^2 - m^2 + i\epsilon)} \\ \langle \bar{\Psi}^T \, \bar{\Psi} \rangle_0 &= C^{-1} \, \langle \Psi \, \bar{\Psi} \rangle_0 = i \tfrac{C^{-1} (\gamma^\mu p_\mu + m)}{(p^2 - m^2 + i\epsilon)} \end{aligned} \tag{28.13}$$

where $C = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}$ is the charge conjugation matrix and we used $C^{-1} \, \Psi = \bar{\Psi}^T$.

The Feynman integrals for the three loops (in terms of 4 component Majorana spinors) are given by

$$\Delta m_1^2 \supset \frac{4!}{2}\,\frac{\lambda^2}{4} \int \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2 - m_1^2 + i\epsilon} + 2\,\frac{\lambda^2}{2} \int \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2 - m_2^2 + i\epsilon} \tag{28.14}$$
$$- \frac{\lambda^2}{2} \int \frac{d^4p}{(2\pi)^4}\,[Tr\{\frac{(\gamma^\mu p_\mu + m)}{(p^2 - m^2 + i\epsilon)}\,\frac{(\gamma^\nu p_\nu + m)}{(p^2 - m^2 + i\epsilon)}\}$$
$$-Tr\{\frac{C^{-1}(\gamma^\mu p_\mu + m)}{(p^2 - m^2 + i\epsilon)}\,\frac{(\gamma^\nu p_\nu + m)C^{-1}}{(p^2 - m^2 + i\epsilon)}\}].$$

Note the combinatoric factors multiplying each graph. Adding the three contributions we see that the quadratic divergent piece of each diagram cancels in the sum. The calculation of the fermion loop could also have been done in terms of 2 component Weyl spinors.

28.3 Solution to Problem 3

There is no SUSY vacuum solution to the F term equations since $F_A = 0$ and $F_C = 0$ cannot be satisfied simultaneously. The solution to $F_B = 0$ gives $2\,a_0\,b_0 + \mu\,c_0 = 0$ where $\langle A\rangle = a_0$, etc. Thus we have a flat direction in the scalar potential with $\frac{c_0}{a_0} = -\frac{2\,b_0}{\mu}$. The scalar potential is now given by

$$V(A,\ B,\ C) = \mu^2\,|B|^2 + |B^2 + M^2|^2. \tag{28.15}$$

We now look for the minimum of the scalar potential. We have

$$\frac{\partial V}{\partial B^*}|_0 = \mu^2\,b_0 + 2(b_0^2 + M^2)b_0^* = 0,\quad \frac{\partial V}{\partial B}|_0 = \mu^2\,b_0^* + 2(b_0^{*2} + M^2)b_0 = 0. \tag{28.16}$$

Multiplying the first equation by b_0, we have

$$(\mu^2 + 2|b_0|^2)\,b_0^2 + 2M^2\,|b_0|^2 = 0. \tag{28.17}$$

Thus b_0^2 is real. Take $b_0^2 = \pm|b_0|^2$. We have

$$\pm\,(\mu^2 + 2|b_0|^2) + 2M^2 = 0 \tag{28.18}$$

implies

$$|b_0|^2 = \mp M^2 - \mu^2/2 \geq 0. \tag{28.19}$$

Let $M^2 > \frac{\mu^2}{2}$, then take the solution $b_0^2 = -|b_0|^2$. We then find

$$|b_0| = \sqrt{M^2 - \frac{\mu^2}{2}} \tag{28.20}$$

and

$$b_0 = \pm i|b_0| \tag{28.21}$$

which gives

$$b_0^2 + M^2 = \frac{\mu^2}{2}. \tag{28.22}$$

We therefore find the vacuum energy at the minimum given by

$$V(a_0,\ b_0,\ c_0) = \mu^2\ (M^2 - \frac{\mu^2}{2}) + \frac{\mu^4}{4}. \tag{28.23}$$

Before calculating the spectrum, let's define the rotated fields

$$G = ((\langle\frac{\partial W}{\partial A}\rangle\ A + \langle\frac{\partial W}{\partial C}\rangle\ C)\ N^{-1} \tag{28.24}$$

$$L = (-\langle\frac{\partial W}{\partial C}\rangle^*\ A + \langle\frac{\partial W}{\partial A}\rangle^*\ C)\ N^{-1} \tag{28.25}$$

with

$$N = (|\langle\frac{\partial W}{\partial A}\rangle|^2 + |\langle\frac{\partial W}{\partial C}\rangle|^2)^{1/2} \equiv V(a_0,\ b_0,\ c_0)^{1/2}. \tag{28.26}$$

If we calculate the vacuum expectation value of G we find

$$\langle G\rangle = ((\langle\frac{\partial W}{\partial A}\rangle\ (a_0 - \theta^2\langle\frac{\partial W}{\partial A}\rangle^*) + \langle\frac{\partial W}{\partial C}\rangle\ (c_0 - \theta^2\langle\frac{\partial W}{\partial C}\rangle^*))\ N^{-1} \tag{28.27}$$

$$= (a_0\ \langle\frac{\partial W}{\partial A}\rangle + c_0\ \langle\frac{\partial W}{\partial C}\rangle)N^{-1} - N\ \theta^2$$

$$= ((b_0^2 + M^2)\ a_0 + \mu b_0\ c_0)N^{-1} - N\ \theta^2$$

$$= a_0(2M^2 - \frac{\mu^2}{2})N^{-1} - N\ \theta^2. \tag{28.28}$$

Similarly, if we calculate the VEV of L we find

$$\langle L\rangle = (-\langle\frac{\partial W}{\partial C}\rangle^* \, (a_0 + \theta^2\langle\frac{\partial W}{\partial A}\rangle^*) + \langle\frac{\partial W}{\partial A}\rangle^* \, (c_0 + \theta^2\langle\frac{\partial W}{\partial C}\rangle^*)) \, N^{-1} \tag{28.29}$$

$$= (-a_0 \, \langle\frac{\partial W}{\partial C}\rangle^* + c_0 \, \langle\frac{\partial W}{\partial A}\rangle^*)N^{-1}$$

$$= -\mu b_0^* \, a_0 + \frac{\mu^2}{2} \, c_0 = -a_0\mu(b_0^* + b_0) = 0. \tag{28.30}$$

Let's now re-write the Lagrangian in terms of the field $G = g_0 + \theta^2 N,\ L$ and $B = B' + b_0$.

Since the transformation from $A,\ C$ to $G,\ L$ is a unitary transformation the Kähler potential becomes

$$\mathscr{K} = G^* \, G + L^* \, L + B'^* \, B' + b_0^* B' + b_0 B'^*. \tag{28.31}$$

and the superpotential becomes

$$\mathscr{W} = A(B'^2 + 2b_0 B' + b_0^2 + M^2) + \mu C(B' + b_0) \tag{28.32}$$

$$= A(B'^2 + 2b_0 B' + \langle\tfrac{\partial W}{\partial A}\rangle) + C(\mu B' + \langle\tfrac{\partial W}{\partial C}\rangle). \tag{28.33}$$

Using

$$A = (\langle\frac{\partial W}{\partial A}\rangle^* \, G - \langle\frac{\partial W}{\partial C}\rangle \, L) \, N^{-1} \tag{28.34}$$

$$C = (\langle\frac{\partial W}{\partial C}\rangle^* \, G + \langle\frac{\partial W}{\partial A}\rangle \, L) \, N^{-1} \tag{28.35}$$

we find

$$\mathscr{W} = (\langle\frac{\partial W}{\partial A}\rangle^* \, G - \langle\frac{\partial W}{\partial C}\rangle \, L) \, N^{-1}(B'^2 + 2b_0 B' + \langle\frac{\partial W}{\partial A}\rangle) \tag{28.36}$$

$$+(\langle\frac{\partial W}{\partial C}\rangle^* \, G + \langle\frac{\partial W}{\partial A}\rangle \, L) \, N^{-1}(\mu B' + \langle\frac{\partial W}{\partial C}\rangle)$$

$$= GN^{-1}(\langle\frac{\partial W}{\partial A}\rangle^*(B'^2 + 2b_0 B' + \langle\frac{\partial W}{\partial A}\rangle) + \langle\frac{\partial W}{\partial C}\rangle^*(\mu B' + \langle\frac{\partial W}{\partial C}\rangle))$$

$$+LN^{-1}(-\langle\frac{\partial W}{\partial C}\rangle(B'^2 + 2b_0 B' + \langle\frac{\partial W}{\partial A}\rangle) + \langle\frac{\partial W}{\partial A}\rangle(\mu B' + \langle\frac{\partial W}{\partial C}\rangle).$$

$$= GN + GN^{-1}(\langle\frac{\partial W}{\partial A}\rangle^*(B'^2 + 2b_0 B') + \langle\frac{\partial W}{\partial C}\rangle^*(\mu B')) \tag{28.37}$$

$$+LN^{-1}(-\langle\frac{\partial W}{\partial C}\rangle(B'^2 + 2b_0 B') + \langle\frac{\partial W}{\partial A}\rangle\mu B').$$

$$= GN + GN^{-1}(\frac{\mu^2}{2}\, B'^2) \tag{28.38}$$

$$+LN^{-1}(-\mu\, b_0 B'^2 + (\frac{3\mu^3}{2} - 2\mu M^2)B')$$

$$\mathscr{W} \equiv GN + \lambda\, (g_0 + \theta^2 N)\, B'^2 + \lambda'\, LB'^2 + m\, L\, B'. \tag{28.39}$$

Note

- $|\langle \frac{\partial W}{\partial G}\rangle|^2 = N^2 = V(a_0,\ b_0,\ c_0)$.
- NO $G\,B'$ or $G\,L$ terms.
- Therefore G is massless and contains the Goldstino.
- B' states have SUSY breaking mass due to coupling to G.

The fermion mass matrix is given by

$$\mathscr{W} \supset m\, L\, B' \Rightarrow \frac{1}{2}\begin{pmatrix} B' & L \end{pmatrix} m_{1/2} \begin{pmatrix} B' \\ L \end{pmatrix} \tag{28.40}$$

where

$$m_{1/2} = \begin{pmatrix} 2\lambda\, g_0 & m \\ m & 0 \end{pmatrix}. \tag{28.41}$$

Scalar mass matrix is given by

$$\frac{1}{2}\begin{pmatrix} B'^* & B' & L^* & L \end{pmatrix} m_0^2 \begin{pmatrix} B' \\ B'^* \\ L \\ L^* \end{pmatrix} \tag{28.42}$$

where

$$m_0^2 = \begin{pmatrix} 4\lambda^2\, g_0^2 + m^2 & \Delta m^2 & 2\lambda\, g_0\, m & 0 \\ \Delta m^2 & 4\lambda^2\, g_0^2 + m^2 & 0 & 2\lambda\, g_0\, m \\ 2\lambda\, g_0\, m & 0 & m^2 & 0 \\ 0 & 2\lambda\, g_0\, m & 0 & m^2 \end{pmatrix}. \tag{28.43}$$

We can now evaluate the supertrace and we find $Str\mathscr{M}^2 = 0$.

28.4 Solution to Problem 4

28.5 SU(5) Representations

First, we calculate how the $\bar{\mathbf{5}}$ transforms. We could do this by taking the totally antisymmetric combination of four **5**s, or we can simply observe that

$$(\mathbf{5}^\alpha)^* \equiv \left(\bar{\mathbf{5}}\right)_\alpha . \tag{28.44}$$

Then

$$\left(\mathbf{5}'\right)^* = (U\mathbf{5})^* = (U)^* \left(\bar{\mathbf{5}}\right) \tag{28.45}$$

Finally, note that

$$(U)^* = \left(\mathbb{1} + i\omega_A T^A + \cdots\right)^* = \mathbb{1} - i\omega_A \left(T^A\right)^T + \cdots . \tag{28.46}$$

So the generators of the anti-fundamental representation are $-\left(T^A\right)^T$.

The index structure of a 10 is given by

$$\mathbf{10}^{\alpha\beta} = \frac{1}{\sqrt{2}}\left[\mathbf{5}^\alpha \mathbf{5}'^\beta - \mathbf{5}^\beta \mathbf{5}'^\alpha\right]. \tag{28.47}$$

Now, given that the transformation law of the **5** is

$$\mathbf{5}'^\alpha = U^\alpha{}_\beta \mathbf{5}^\beta = \left\{\delta^\alpha{}_\beta + i\omega^A (T_A)^\alpha{}_\beta + \mathcal{O}(\omega^2)\right\} \mathbf{5}^\beta, \tag{28.48}$$

we find :

$$\begin{aligned} \mathbf{10}^{\alpha\beta} \to \mathbf{10}'^{\alpha\beta} &= \left(\delta^\alpha{}_\gamma + i\omega^A (T_A)^\alpha{}_\gamma\right)\left(\delta^\beta{}_\delta + i\omega^A (T_A)^\beta{}_\delta\right)\mathbf{10}^{\gamma\delta}, \\ &= \left\{\delta^\alpha{}_\gamma \delta^\beta{}_\delta + i\omega^A \left[\delta^\alpha{}_\gamma (T_A)^\beta{}_\delta + \delta^\beta{}_\delta (T_A)^\alpha{}_\gamma\right]\right\}\mathbf{10}^{\gamma\delta}. \end{aligned} \tag{28.49}$$

Therefore

$$\mathbf{10}'^{\alpha\beta} \equiv \left\{\delta^\alpha{}_\gamma \delta^\beta{}_\delta + i\omega^A T^{\alpha\beta}_{A\gamma\delta}\right\}\mathbf{10}^{\gamma\delta} \tag{28.50}$$

where

$$T^{\alpha\beta}_{A\gamma\sigma} \equiv \left[\delta^\alpha{}_\gamma (T_A)^\beta{}_\delta + \delta^\beta{}_\delta (T_A)^\alpha{}_\gamma\right] \tag{28.51}$$

are the generators acting on the **10**.

Now evaluate the generators on the adjoint representation. We have the field Σ in the adjoint representation given by a traceless matrix

$$\Sigma^{\alpha}{}_{\beta} = \sum_{A} \Sigma^{A} T_{A}{}^{\alpha}{}_{\beta}. \tag{28.52}$$

Then under an $SU(5)$ transformation we have

$$\Sigma'^{\alpha}{}_{\beta} = U^{\alpha}{}_{\gamma}\, U^{*\beta}{}_{\delta}\, \Sigma^{\gamma}{}_{\delta} = U^{\alpha}{}_{\gamma}\, \Sigma^{\gamma}{}_{\delta}\, U^{\dagger\delta}{}_{\beta}. \tag{28.53}$$

Under an infinitesimal transformation we have

$$\Sigma'^{\alpha}{}_{\beta} = \left(\mathbb{1} + i\omega_A T^A + \cdots\right)\, \Sigma^{\gamma}{}_{\delta}\, \left(\mathbb{1} - i\omega_A \left(T^A\right) + \cdots\right) = \Sigma^{\alpha}{}_{\beta} + iT_A{}^{\alpha\gamma}{}_{\beta\delta}\, \Sigma^{\gamma}{}_{\delta} \tag{28.54}$$

with

$$T_A{}^{\alpha\gamma}{}_{\beta\delta}\, \Sigma^{\gamma}{}_{\delta} = [T_A,\ \Sigma]^{\alpha}{}_{\beta}. \tag{28.55}$$

28.6 Higgsing SU(5)

The adjoint representation is the smallest representation which will work for higgsing SU(5), however, larger representations can be used (and MUST be used to get realistic models). The reason that we need *at least* a **24** is easy to see—once we assigned fermions to the **10** and $\bar{\mathbf{5}}$, we defined a charge operator. Specifically, there is no hypercharge neutral component of the $\bar{\mathbf{5}}$ or the **10**, so giving any of the components of those representations a VEV breaks hypercharge.

To calculate the gauge boson masses, we need to write down a Lagrangian that tells us how they interact with the Higgs:

$$\mathscr{L}_{\text{higgs}} = \operatorname{Tr}\left[\left(D_\mu \Sigma\right)^\dagger D^\mu \Sigma\right] + V(\Sigma^\dagger \Sigma). \tag{28.56}$$

Since we're only really interested in the mass terms, we can see that

$$\begin{aligned} \mathscr{L}_{\text{higgs}} &= \text{stuff} + g^2 A^A_\mu A^{\mu B} \operatorname{Tr}\left\{[T_A, \Sigma]\,.\,([T_B, \Sigma])^\dagger\right\}, \\ &= \text{stuff} - g^2 A^A_\mu A^{\mu B} \operatorname{Tr}\left\{[T_A, \Sigma]\,.\,[T_B, \Sigma]\right\}. \end{aligned} \tag{28.57}$$

where we made the last replacement because $\Sigma^\dagger = \Sigma, T_B^\dagger = T_B$. In order to calculate the trace, we'll need to know the form of at least one of the SU(5) generators. Luckily, two of them are listed in the notes, Eq. (5.19). We'll take

$$T_A = \frac{1}{2}\begin{pmatrix} & & & 1 & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix}, \tag{28.58}$$

where the factor of $\frac{1}{2}$ is to ensure that the generators are properly normalized. Then

$$(\Sigma.T_A - T_A.\Sigma)^2 = -V^2 \ \ \text{diag}(\frac{25}{16}, 0, 0, \frac{25}{16}, 0). \tag{28.59}$$

This means

$$-g^2 A^A_\mu A^{\mu B} \ \text{Tr}\ \{[T_A, \Sigma] . [T_B, \Sigma]\} = \frac{25}{8} g^2 V^2 \delta_{AB} A^A_\mu A^{\mu B}, \tag{28.60}$$

which, for a canonically normalized gauge field, means that we have mass

$$m_A^2 = \frac{25}{4} g^2 V^2. \tag{28.61}$$

Flipped SU(5)

The easiest way to see that the gauge group of the flipped SU(5) "GUT" (or $\widetilde{\text{SU}(5)}$, for short) is to check the decomposition of the GUT representations under the SM gauge group. In $\widetilde{\text{SU}(5)}$ we interchange the fields $\bar{u} \to \bar{d}$ and $\bar{\nu} \to \bar{e}$. Slansky tells us

$$\bar{\mathbf{5}} \to (\bar{\mathbf{3}}, 1)_{2/3} + (1, \mathbf{2})_{-1}. \tag{28.62}$$

So it seems pretty natural to just identify the U(1) generator which lives in SU(5) with the hypercharge generator. In the case of $\widetilde{\text{SU}(5)}$, we can no longer make this identification, because the hypercharge of $\bar{u}$ is not 2/3. This means that the actual hypercharge must be (at least) a linear combination of two U(1) generators—one of which is embedded in SU(5) and one of which is not.

To calculate the definition of hypercharge in terms of the $U(1)_X$ and $U(1)_{\tilde{Y}}$, we first note that the standard definition of electric charge is $Q = T_3 + Y/2$. The new definition, in terms of X and $\tilde{Y}$ quantum numbers is

$$Q = T_3 + a\tilde{Y} + bX. \tag{28.63}$$

Now we can calculate the charge of the electron:

$$Q = -\frac{1}{2} + a(-1) + b(-3) = -1 \Rightarrow a + 3b = \frac{1}{2}, \tag{28.64}$$

and of the $\bar{u}$:

$$Q = 0 + a\left(\frac{2}{3}\right) + b(-3) = -\frac{2}{3} \Rightarrow 2a - 9b = -2. \tag{28.65}$$

This gives us

$$\Rightarrow Q = T_3 - \frac{1}{10}\tilde{Y} + \frac{1}{5}X. \tag{28.66}$$

Comparing to the familiar definition of hypercharge, we find

$$Y = -\frac{1}{5}\tilde{Y} + \frac{2}{5}X. \tag{28.67}$$

Once we have this definition of hypercharge, it's relatively easy to assign the fermions to the **10**. So, for example, we know that all of the components of the **10** have $X = 1$. The q states still live in the same place (as expected), because (for the up quark, for example)

$$Q = \frac{1}{2} - \frac{1}{10}\frac{1}{3} + \frac{1}{5}(+1) = \frac{2}{3}. \tag{28.68}$$

The $\bar{d}$ quarks "flip" places with the $\bar{u}$ quarks:

$$Q = 0 - \frac{1}{10}\left(-\frac{4}{3}\right) + \frac{1}{5}(+1) = \frac{1}{3}. \tag{28.69}$$

The singlet has

$$Q = 0 - \frac{1}{10}(+2) + \frac{1}{5}(+1) = 0. \tag{28.70}$$

This means that the $\widetilde{\mathrm{SU}(5)}$ model *requires* a right handed neutrino! (This is not really surprising, if you consider that the $\widetilde{\mathrm{SU}(5)}$ is just another embedding of SU(5) into SO(10).) The only state that is left is the anti-electron, $\bar{e}$, which lives in the singlet, which must have $X = 5$:

$$Q = 0 - \frac{1}{10}(0) + \frac{1}{5}(+5) = 1. \tag{28.71}$$

Finally, we can look at symmetry breaking in this model. In fact, we don't need the adjoint representation any more—the reason is easy to see. Because the positron and the right handed neutrino have "flipped" places in the **10** and the 1, the **10** now has a hypercharge-neutral component, which can get a VEV. This means that we can take $\Sigma = \mathbf{10}$ with

$$\langle \Sigma \rangle = V \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{28.72}$$

This model suffers from the absence of a prediction for gauge coupling unification and also some of the fermion mass predictions which plague minimal SU(5).

28.7 Solution to Problem 5

The superpotential for SUSY $SU(5)$ is given by

$$\mathscr{W} = \frac{\lambda}{3}\, Tr(\Sigma^3) - \frac{M}{2}\, Tr(\Sigma^2) + \bar{H}\,(\lambda'\,\Sigma - M')\,H. \tag{28.73}$$

The vacuum solution is given by $\Sigma_0 = \frac{\sqrt{60}}{\lambda}\, M T_{24}$ and $M' = \frac{3\lambda'}{\lambda}\, M$. Given the definitions of

$$T_{24} = \sqrt{\frac{3}{5}} \left(\begin{array}{ccc|cc} -1/3 & 0 & 0 & & \\ 0 & -1/3 & 0 & & 0 \\ 0 & 0 & -1/3 & & \\ \hline & & & 1/2 & 0 \\ & 0 & & 0 & 1/2 \end{array}\right) \tag{28.74}$$

and defining $T_9,\ T_{10}$ by the equations

$$T_9 = \frac{1}{2} \left(\begin{array}{ccc|cc} & & & 1 & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ \hline 1 & 0 & 0 & & \\ 0 & 0 & 0 & & 0 \end{array}\right), \quad T_{10} = \frac{1}{2} \left(\begin{array}{ccc|cc} & & & -i & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ \hline i & 0 & 0 & & \\ 0 & 0 & 0 & & 0 \end{array}\right) \tag{28.75}$$

we can derive the useful identities

$$[T_{24}, T_9] = -\frac{i}{2}\sqrt{\frac{5}{3}}\, T_{10}, [T_{24}, T_{10}] = \frac{i}{2}\sqrt{\frac{5}{3}}\, T_9 \tag{28.76}$$

$$\{T_{24}, T_9\} = \frac{1}{2\sqrt{15}}\, T_9, \{T_{24}, T_{10}\} = \frac{1}{2\sqrt{15}}\, T_{10} \tag{28.77}$$

with similar relations for $T_A,\ A = 11 - 20$.

The scalar potential is given by

$$V = |\frac{\partial \mathscr{W}}{\partial \Sigma}|^2 + |\frac{\partial \mathscr{W}}{\partial H}|^2 + |\frac{\partial \mathscr{W}}{\partial \bar{H}}|^2 + \frac{1}{2}\sum_A D_A^2 \tag{28.78}$$

where $D_A = g_5 Tr(\Sigma^\dagger\ [T_A, \Sigma])$. The field Σ can be written in terms of the 24 normalized complex scalar fields Σ_A via the equation

$$\Sigma = \sqrt{2}\sum_A T_A\ \Sigma_A. \tag{28.79}$$

Let us first show that $D_A(\Sigma_0) = 0$. We have

$$D_A|_{\Sigma_0} = g_5 Tr(\Sigma_0^\dagger [T_A, \Sigma_0]) = g_5 \frac{60}{\lambda^2} M^2 Tr(T_{24}[T_A, T_{24}]). \tag{28.80}$$

BUT

$$[T_A, T_{24}] = 0, \ \text{ for } \ A = 1, \cdots, 8; 21, 22, 23; 24 \tag{28.81}$$

and

$$Tr(T_{24}T_A) = 0 \ \text{ for } \ A = 9 - 20. \tag{28.82}$$

Therefore $D_A|_{\Sigma_0} = 0$.

To calculate the scalar masses we need the quadratic terms in the scalar potential after shifting the Σ field by $\Sigma = \Sigma_0 + \tilde{\Sigma}$. Let's first consider the mass terms coming from the F terms scalar potential. We have

$$\begin{aligned} V_2(\tilde{\Sigma}) &= Tr((\frac{\partial \mathscr{W}}{\partial \Sigma})^\dagger ((\frac{\partial \mathscr{W}}{\partial \Sigma}) \\ &= Tr\{[\lambda(\{\Sigma_0, \tilde{\Sigma}\}^\dagger - (Tr\{\Sigma_0, \tilde{\Sigma}\})^\dagger\ \frac{\mathscr{I}}{5}) - M\tilde{\Sigma}^\dagger] \\ &\quad [\lambda(\{\Sigma_0, \tilde{\Sigma}\} - (Tr\{\Sigma_0, \tilde{\Sigma}\})\ \frac{\mathscr{I}}{5}) - M\tilde{\Sigma}]\} \end{aligned} \tag{28.83}$$

$$= M^2 Tr\{(\sqrt{60}[\{T_{24}, \tilde{\Sigma}\}^\dagger - Tr\{T_{24}, \tilde{\Sigma}\}^\dagger \frac{\mathscr{I}}{5}] - \tilde{\Sigma}^\dagger)$$
$$(\sqrt{60}[\{T_{24}, \tilde{\Sigma}\} - Tr\{T_{24}, \tilde{\Sigma}\} \frac{\mathscr{I}}{5}] - \tilde{\Sigma})\}.$$

Using the identity

$$Tr\{T_{24}, \tilde{\Sigma}\} = \sqrt{2} \sum_A Tr\{T_{24}, T_A\} \Sigma_A = \sqrt{2} \Sigma_{24} \tag{28.84}$$

we find

$$\begin{aligned} V_2(\tilde{\Sigma}) &= 2M^2 Tr\{[-5 \sum_{A=1,\cdots,8} T_A \Sigma_A^\dagger + 5 \sum_{A=21,22,23} T_A \Sigma_A^\dagger + T_{24} \Sigma_{24}^\dagger] \\ &\quad \times [-5 \sum_{A=1,\cdots,8} T_A \Sigma_A + 5 \sum_{A=21,22,23} T_A \Sigma_A + T_{24} \Sigma_{24}]\} \\ &= 25M^2 \sum_{A=1,\cdots,8} \Sigma_A^\dagger \Sigma_A + 25M^2 \sum_{A=21,22,23} \Sigma_A^\dagger \Sigma_A + M^2 \; \Sigma_{24}^\dagger \Sigma_{24}. \end{aligned} \tag{28.85}$$

Thus we have $m_8 = 5M = m_3$ for the color octet and $SU(2)$ triplet and $m_1 = M$ for the SM singlet field.

Note, the fields $\Sigma_A, \;\; A = 9-20$ are, at the moment, massless. Now consider the D term potential. The quadratic mass terms are given by

$$V_2' = \frac{g_5^2}{2} \sum_A [Tr((\Sigma^\dagger - \Sigma)[T_A, \Sigma_0]]^2 = \frac{g_5^2 60M^2}{2\lambda^2} \sum_A [Tr((\Sigma^\dagger - \Sigma)[T_A, T_{24}]]^2 \tag{28.86}$$
$$= \frac{g_5^2 60M^2}{\lambda^2} \sum_A (\Sigma_B^\dagger - \Sigma_B) Tr(T_B[T_A, T_{24}])(\Sigma_C^\dagger - \Sigma_C) Tr(T_C[T_A, T_{24}]).$$

Defining $\Sigma_A = (\Sigma_{Ar} + i\Sigma Ai)/\sqrt{2}$ we have $\Sigma_A^\dagger - \Sigma_A = -\sqrt{2} i \Sigma_{Ai}$. Using the commutation identities, Eq. (28.76), we obtain

$$V_2' = \frac{1}{2} m_X^2 \sum_{A=9-20} \Sigma_{Ai}^2 \tag{28.87}$$

with $m_X^2 = 25 g_5^2 \frac{M^2}{\lambda^2}$. m_X is identical to the mass for the massive $SU(5)$ gauge bosons. In Problem 3 we took $\Sigma_0 = V \mathrm{diag}(1, 1, 1, -3/2, -3/2)$. This is identical to the symmetry breaking VEV here if we take $V = -\frac{2M}{\lambda}$.

The color triplet Higgs scalars have mass $m_T = m_{\bar{T}} = 5\frac{\lambda'}{\lambda} M$ which is equal to the Dirac mass of the color triplet Higgsinos. Finally, the fermions in $\tilde{\Sigma}$ obtain mass from the superpotential and also from the gaugino-$\tilde{\Sigma}$ couplings. The gauginos mix with the $\tilde{\Sigma}_A, \quad A = 9 - 20$ to form Dirac gauginos with mass equal to m_X. On the other hand the rest of the $\tilde{\Sigma}$ fermions obtain mass from the superpotential given by

$$\begin{aligned}
\frac{\partial^2 \mathscr{W}}{\partial \Sigma_A \partial \Sigma_B}|_{\Sigma_0} &= 2\lambda Tr(T_A\{T_B, \Sigma_0\}) - M\delta_{AB} \qquad (28.88)\\
&= M[2\sqrt{60} Tr(T_A\{T_B, T_{24}\}) - \delta_{AB}]\\
&= M[-5\delta_{AB} \ \ (\text{for} \ \ A, B = 1, \cdots, 8) + 5\delta_{AB} \ \ (\text{for} \ \ A, B = 21, 22, 23)\\
&\quad +\delta_{AB} \ (\text{for}\ A, B = 24)].
\end{aligned}$$

Thus again we have $m_8 = m_3 = 5M$ and $m_1 = M$.

28.8 Solution to Problem 6

$$\left(\frac{3}{5} - \frac{8}{5}\ \sin^2\theta_W(M_Z)\right)\alpha_{EM}(M_Z)^{-1} = (\frac{b_2 - b_1}{2\pi})\ln(\frac{M_G}{M_Z}) \qquad (28.89)$$

which we use to solve for M_G. Then we use

$$\alpha_G^{-1} = \sin^2\theta_W(M_Z)\ \alpha_{EM}(M_Z)^{-1} + \frac{b_2}{2\pi}\ln(\frac{M_G}{M_Z}) \qquad (28.90)$$

to solve for α_G. We can then predict the value for the strong coupling using

$$\alpha_3(M_Z)^{-1} = \alpha_G^{-1} - \frac{b_3}{2\pi}\ln(\frac{M_G}{M_Z}). \qquad (28.91)$$

We use $\sin^2\theta_W = 0.23,\ \alpha_{EM}^{-1} = 128, \quad M_Z = 91.2$ GeV.

For SUSY we have

$$\mathbf{b}_{SUSY} = (-2N_{fam} - \frac{3}{5}N_{(H_u+H_d)}, 6 - 2N_{fam} - N_{(H_u+H_d)}, 9 - 2N_{fam}) \qquad (28.92)$$

where $N_{(H_u+H_d)}$ is the number of pairs of Higgs doublets.

Case 1) $N_{fam} = 4, \ \ N_{(H_u+H_d)} = 1; \quad \mathbf{b}_{SUSY} = (-43/5, -3, 1)$ We find

$$M_G = 2.6 \times 10^{16}\ \text{GeV}, \quad \alpha_G^{-1} = 13.5, \quad \alpha_s(M_Z) = 0.122. \qquad (28.93)$$

Case 2) $N_{fam} = 3, \ \ N_{(H_u+H_d)} = 2; \quad \mathbf{b}_{SUSY} = (-36/5, -2, 3)$ We find

$$M_G = 3.6 \times 10^{17}\ \text{GeV}, \quad \alpha_G^{-1} = 17.9, \quad \alpha_s(M_Z) = 0.77. \qquad (28.94)$$

28.9 Solution to Problem 7

The quaternionic group has five conjugacy classes given by $\{e\}$, $\{a_1\}$, $\{a_2, a_3\}$, $\{a_4, a_5\}$, $\{a_6, a_7\}$. You should have found that the isomorphism between the group elements is given by, $a_i \leftrightarrow D_{a_i}$ for all i and $e \leftrightarrow D_e$.

To prove that $\overrightarrow{X} \cdot \overrightarrow{Y} = x_1\, y_2 - x_2\, y_1$ is invariant, you just need to take one element D_{a_i} from each conjugacy class and given the transformed doublet $\overrightarrow{X}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = D_{a_i} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = D_{a_i} \overrightarrow{X}$ and similarly for $\overrightarrow{Y}$, show that $\overrightarrow{X}' \cdot \overrightarrow{Y}' = \overrightarrow{X} \cdot \overrightarrow{Y}$.

28.10 Solution to Problem 8

First find the extremum of the scalar potential. Evaluating the term

$$\phi_-^* \frac{\partial V}{\partial \phi_+} + \phi_+^* \frac{\partial V}{\partial \phi_-} = 0 \tag{28.95}$$

we find

$$\phi_+\, \phi_- = \phi_+^*\, \phi_-^* = \frac{M^2(|\phi_+|^2 + |\phi_-|^2)}{(|\phi_+|^2 + |\phi_-|^2 + m_+^2 + m_-^2)}. \tag{28.96}$$

Evaluating the term

$$\phi_+ \frac{\partial V}{\partial \phi_+} - \phi_- \frac{\partial V}{\partial \phi_-} = 0 \tag{28.97}$$

we find

$$(|\phi_+|^2 - |\phi_-|^2) \simeq -\frac{\delta\, m^2}{2g^2}. \tag{28.98}$$

At the scale M the spectrum is approximately supersymmetric with small corrections of order $m/M \ll 1$. In this note, we neglect corrections of order m/M. Shifting the scalar fields by their expectation values we have

$$\phi_\pm = \langle \phi_\pm \rangle + \phi_\pm'. \tag{28.99}$$

We take

$$\langle \phi_+ \rangle \approx \langle \phi_+ \rangle \approx M \tag{28.100}$$

real. The VEVs spontaneously break the $U(1)$ gauge symmetry at the scale, M.

Due to the term in the scalar potential

$$V \supset |\phi_+ \phi_- - M^2|^2 \tag{28.101}$$

the field $\frac{\phi'_+ + \phi'_-}{\sqrt{2}}$ obtains mass, $\sqrt{2}M$. The orthogonal linear combination, $\frac{\phi'_+ - \phi'_-}{\sqrt{2}}$, remains massless. Its imaginary component is the goldstone boson of spontaneous $U(1)$ breaking and is eaten by the $U(1)$ gauge bosons. Its real part, $\left(\frac{\phi_{+r} - \phi_{-r}}{\sqrt{2}}\right)$, obtains mass from the D-term.

Consider the term in the scalar potential due to the $U(1)$ D-term. We have

$$V_D \simeq \frac{g^2}{2}\left(-\frac{\delta\, m^2}{2g^2} + 2\langle \phi_+ \rangle \left(\frac{\phi_{+r} - \phi_{-r}}{\sqrt{2}}\right) + \sum_i q_i |\phi_i|^2\right)^2 . \tag{28.102}$$

Note, contained in this term there exists a tadpole term for the field $\left(\frac{\phi_{+r} - \phi_{-r}}{\sqrt{2}}\right)$. However there cannot be such a tadpole term at the extremum of the potential. We now show that this tadpole term is cancelled exactly from a similar term coming from the soft scalar mass terms. We have

$$m_+^2 |\phi_+|^2 + m_-^2 |\phi_-|^2 = m^2(|\phi_+|^2 + |\phi_-|^2) + \frac{\delta m^2}{2}(|\phi_+|^2 - |\phi_-|^2). \tag{28.103}$$

Only the second term gives a tadpole for the field $\left(\frac{\phi_{+r} - \phi_{-r}}{\sqrt{2}}\right)$. It gives

$$\delta\, m^2 \langle \phi_+ \rangle \left(\frac{\phi_{+r} - \phi_{-r}}{\sqrt{2}}\right). \tag{28.104}$$

Therefore the sum of the two tadpole terms vanish.

As a result of the D-term we obtain the following contributions to the light scalar potential. We have

$$\begin{aligned} V_D \supset & -\frac{\delta\, m^2}{2} \sum_i q_i\, |\phi_i|^2 + \frac{g^2}{2}\left(\sum_i q_i |\phi_i|^2\right)^2 \\ & + \left(\sum_i q_i\, |\phi_i|^2\right) 2g^2\, \langle \phi_+ \rangle \left(\frac{\phi_{+r} - \phi_{-r}}{\sqrt{2}}\right) \\ & + \frac{4g^2 \langle \phi_+ \rangle^2}{2}\left(\frac{\phi_{+r} - \phi_{-r}}{\sqrt{2}}\right)^2 . \end{aligned} \tag{28.105}$$

The last term gives mass $\bar{M} = 2g\langle\phi_+\rangle \simeq 2gM$ to the real field. The first term corresponds to a D-term mass for the light fields. The second term corresponds to the quartic scalar term from the $U(1)$ D term. However, the $U(1)$ gauge symmetry is broken at the scale M, so how is it possible to have the quartic scalar coupling in the low energy theory? In fact, it decouples from the low energy theory, as we now show. The actual quartic term in the effective low energy theory has two parts. The first part given by the second term and a second part due to the tree level exchange of the real field, $\left(\frac{\phi_{+r}-\phi_{-r}}{\sqrt{2}}\right)$.

$$V_D^{eff} = -\frac{\delta\, m^2}{2} \sum_i q_i\, |\phi_i|^2 + \frac{g^2}{2} \left(\sum_i q_i|\phi_i|^2\right)^2 \tag{28.106}$$

$$+2\, \frac{(g^2\, \langle\phi_+\rangle)^2}{p^2 - \bar{M}^2} \left(\sum_i q_i\, |\phi_i|^2\right)^2 \tag{28.107}$$

$$= -\frac{\delta\, m^2}{2} \sum_i q_i\, |\phi_i|^2 + \left(1 + \frac{\bar{M}^2}{p^2 - \bar{M}^2}\right) \left(\sum_i q_i|\phi_i|^2\right)^2$$

$$\simeq -\frac{\delta\, m^2}{2} \sum_i q_i\, |\phi_i|^2 \quad (\text{for } p^2/\bar{M}^2 \ll 1).$$

These correspond to the two terms in Fig. 28.2.

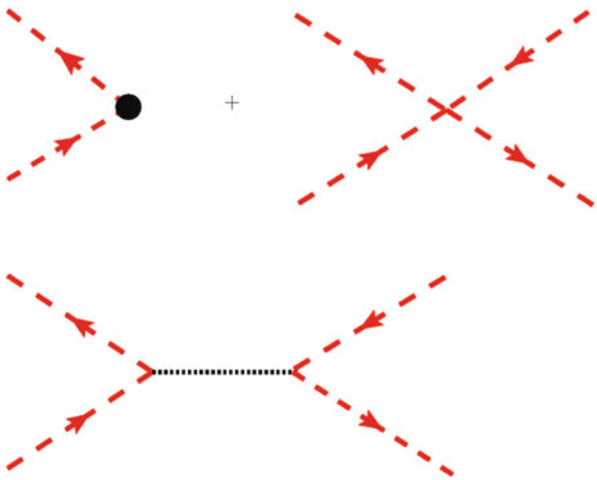

Fig. 28.2 The *top figure* is the tree level D term spitting out two scalars plus the tree level quartic coupling. The *bottom figure* is the exchange of the real scalar in the massive gauge multiplet

In this problem, we have shown that soft SUSY breaking $U(1)$ quartic terms decouple when the $U(1)$ symmetry is spontaneously broken at a scale, $M \gg m$, where m is the effective low energy SUSY breaking scale. The D term, on the other hand, remains in the effective low energy theory.

28.11 Solution to Problem 9

Case 1: $\rho_\phi \ll \rho_r$

Let us first consider the case when the universe is radiation dominated. In this era, the scale factor is given by $a^2 = Ct$, where $C = \frac{2\,T_0^2}{\sqrt{3}m_{Pl}^2}$. As initial conditions we take $C = 1$. The Hubble parameter is then $H = (2t)^{-1}$ and the equation of motion is

$$\ddot{\phi} + \frac{3}{2t}\dot{\phi} + m^2\phi = 0\,. \tag{28.108}$$

Using `Mathematica`, the analytic solution for this equation is determined to be

$$\phi(t) = \frac{2^{1/4}m^{3/4}\Gamma\left(\frac{5}{4}\right)J_{\frac{1}{4}}(mt)}{t^{1/4}}, \tag{28.109}$$

where $J_n(z)$ is the Bessel function of the first kind and $\Gamma(n)$ is the Euler gamma function.

In order to fully understand this result we use the trial solution $\phi = \phi_0\, e^{i\omega(t)}$ and solve for ω. In the limit that $|i3H\dot{\omega}| \gg (\dot{\omega})^2$ and initial conditions $\phi_0 = m$ and $\dot{\omega}(0) = 0$, the equation of motion becomes

$$i\ddot{\omega} + \frac{3i}{2t}\dot{\omega} + m^2 = 0. \tag{28.110}$$

Initially, $H \gg m$. Since this causes the oscillations of ϕ to be over-damped, we want to initially ignore $\ddot{\omega}$. However, because $\dot{\omega}(0) = 0$, let us keep $\ddot{\omega}$ for very early times and we find the solution $\omega(t) = im^2t^2/5$. Thus for very early times we have

$$\phi(t) \simeq \phi_0\, e^{-\frac{m^2}{5}t^2}. \tag{28.111}$$

This continues until $3\,H \sim m$.

When $3H \sim m$, the damping is non-negligible and we now ignore $\ddot{\omega}$. Setting $3H = m$, the equation of motion becomes

$$-\dot{\omega}^2 + im\dot{\omega} + m^2 = 0. \tag{28.112}$$

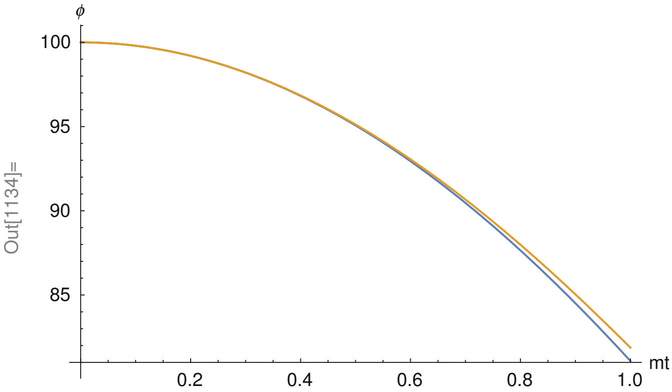

Fig. 28.3 The exact solution and approximate piecewise solution for $\phi(t)$ at early times are shown by the *blue and orange lines*, respectively

The solutions in this regime are

$$\omega_{\pm}(t) = \frac{imt}{2} \pm \frac{\sqrt{3}mt}{2} \tag{28.113}$$

$$\rightarrow \phi(t) = \phi_0' e^{-\frac{1}{2}mt} \cos \frac{\sqrt{3}}{2} mt, \tag{28.114}$$

where the choice of ω_+ or ω_- doesn't matter since we take the real part of $e^{i\omega(t)}$ and cosine is an even function. The coefficient ϕ_0' is determined by matching the solutions of the two regimes discussed so far at the crossover. Taking the time of the crossover to be $t = 1/m$, we find $\phi_0' = me^{-1/3}$.

We plot the exact solution and the piecewise constructed solution in Fig. 28.3 for very early times. Note, that the energy in the ϕ field decreases very slowly between the initial time, $t \sim 1/m_{Pl}$, and late times when $t \sim 1/m$, since the field is overdamped due to Hubble friction.

Case 2: $3H < m$

Finally, we want to discuss late times when $3H \ll m$. Here, for the moment, we ignore the Hubble parameter in Eq. (28.110) and obtain for this regime

$$i\ddot{\omega} - \dot{\omega}^2 + m^2 = 0. \tag{28.115}$$

This equation is second order and therefore has two solutions. One solution can easily be seen to be $\omega = mt$. To obtain the full behavior at late times, let us construct a solution by multiplying the simple oscillatory behavior, $\cos mt$, by some power of the scale factor, $a(t)$. Let us now quantify what would be considered "late times" and determine the behavior of the scalar field in this limit. The continuity equation for ρ_ϕ is given by

$$\dot{\rho}_\phi = -3H(\rho_\phi + p_\phi). \tag{28.116}$$

The pressure can determined from the stress-energy tensor,

$$p_\phi = T_{ii} = +\mathscr{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2. \tag{28.117}$$

Due to the simple oscillatory behavior of $\phi(t)$, we can take $\langle p_\phi \rangle = 0$ when averaged over many cycles. We therefore expect for the energy density to decrease as a^{-3}. Since the universe is radiation dominated,

$$\left(\frac{a_{late}}{a(t)}\right)^2 = \frac{t_{late}}{t}, \tag{28.118}$$

where the subscript "late" denotes the time at which the relation $3H \ll m$ is valid. If the energy density drops like a^{-3}, then ϕ drops like $a^{-3/2}$ and we find

$$\phi(t) = \phi_0'' \cos mt \left(\frac{t_{late}}{t}\right)^{3/4}. \tag{28.119}$$

We can determine ϕ_0'' and t_{late} by choosing the crossover to this final regime to be when $3H/m = 1/e$. This gives $t_{late} = \frac{3e}{2m}$ and ϕ_0'' is determined by inserting t_{late} into Eq. (28.114). We plot the exact solution and the piecewise constructed solution in Fig. 28.4 for all times. After times $t \sim 1/m$, the field oscillates and loses energy as pressureless matter.

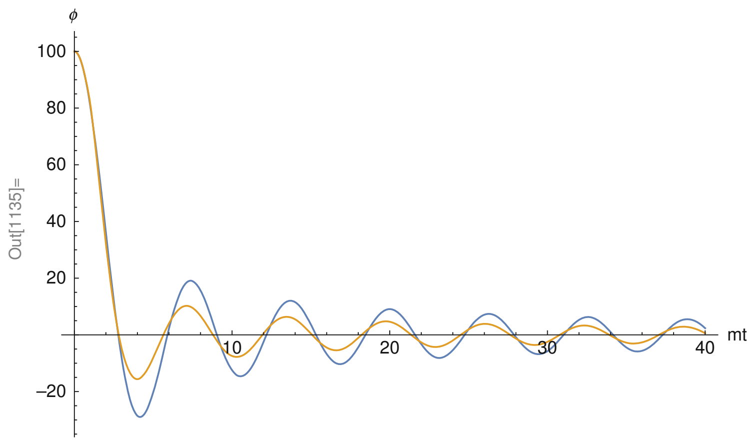

Fig. 28.4 The exact solution and approximate piecewise solution for $\phi(t)$ for all times are shown by the *blue and orange lines*, respectively

At even later times the total energy density is dominated by the energy density stored in the scalar field, the Hubble parameter can be expressed solely in terms of ϕ and $\dot{\phi}$.

$$H^2 = \frac{\rho_\phi}{3m_{Pl}^2} \rightarrow H = \left[\frac{\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2}{3m_{Pl}^2}\right]^{1/2} \tag{28.120}$$

The energy in the ϕ field still satisfies $\rho_\phi \sim 1/a^3$. The only difference is the dependence of the scale factor, a, on time. During the matter dominated epoch we have $a(t) \sim t^{2/3}$.

28.12 Solution to Problem 10

The solutions to these problems uses the identity

$$e^{i\theta} = \cos\theta + i\sin\theta = 1 + \sum_{k=1}^{\infty}\frac{(i\theta)^{2k}}{(2k)!} + \sum_{k=1}^{\infty}\frac{(i\theta)^{2k+1}}{(2k+1)!}. \tag{28.121}$$

(a) Using $T^3\ T^{\pm} = T^{\pm}\ (T^3 \pm 1)$ and $(T^3)^2\ T^{\pm} = T^{\pm}\ (T \pm 1)^2$ and

$$e^{i\frac{y}{R}T^3} = 1 - (T^3)^2 + (T^3)^2\ \cos(\frac{y}{R}) + iT^3\ \sin(\frac{y}{R}), \tag{28.122}$$

we have

$$e^{i\frac{y}{R}T^3}\ T^{\pm} = T^{\pm}(1-(T^3\pm1)^2+(T^3\pm1)^2\ \cos(\frac{y}{R})+i(T^3\pm1)\ \sin(\frac{y}{R})) = T^{\pm}\ e^{i\frac{y}{R}(T^3\pm1)}. \tag{28.123}$$

Therefore we have

$$e^{i\frac{y}{R}T^3}\ T^{\pm}\ e^{-i\frac{y}{R}T^3} = e^{\pm i\frac{y}{R}}\ T^{\pm}. \tag{28.124}$$

(b) We also have that

$$e^{i\pi T^3} = 1 - (T^3)^2 + (T^3)^2\ \cos(\pi) + iT^3\ \sin(\pi) = diag(-1,-1,1). \tag{28.125}$$

28.13 Solution to Problem 11

1. (a)

$$T = \exp(-i\frac{3}{2}\pi(B{-}L)) \quad \text{with} \quad (B{-}L) = \frac{2}{3}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \equiv \frac{2}{3}\alpha. \tag{28.126}$$

Note, $\alpha^{2n} = diag(1,1,1,1,1,1,0,0,0,0) \equiv \beta$ and $\alpha^{2n+1} = \alpha$. Thus

$$\begin{aligned} T = \exp(-i\pi\alpha) &= \sum_{n=0}^{\infty} \frac{(-i\pi\alpha)^n}{n!} \qquad (28.127) \\ &= \sum_{n=0}^{\infty} \frac{(-i\pi\alpha)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-i\pi\alpha)^{2n+1}}{(2n+1)!} \\ &= 1 + \beta \sum_{n=1}^{\infty} \frac{(-1)^n(\pi)^{2n}}{(2n)!} - i\alpha \sum_{n=0}^{\infty} \frac{(-1)^n(\pi)^{2n+1}}{(2n+1)!} \\ &= 1 + \beta\ (cos\pi - 1) - i\alpha\ \sin\pi = 1 - 2\beta \\ &= diag(-1,-1,-1,-1,-1,-1,1,1,1,1). \end{aligned}$$

(b)

$$T = \exp(i3\pi Y) \quad \text{with} \quad 3Y = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \equiv \alpha. \tag{28.128}$$

However

$$T = e^{i\pi\alpha} = \begin{pmatrix} +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{28.129}$$

28.14 Solution to Problem 12

We want to evaluate the one-loop renormalization of the gauge coupling constants in the orbifold GUT theory with the sum over all KK modes. We start with the vacuum polarization tensor which contributes to the β function. Consider the vacuum polarization of the gauge fields with fermions in the loop. The vacuum polarization is given by

$$\Pi_{\mu\nu}(k) = -\sum_{n=-\infty}^{\infty} g^2 \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\left(\gamma_\mu \frac{1}{\not q - m_n}\gamma_\nu \frac{1}{\not k + \not q - m_n}\right), \tag{28.130}$$

where m_n is the mass of the n Kaluza-Klein excitation. The negative sign is due to the fermion loop.

From Ward identity

$$\begin{aligned} \Pi_{\mu\nu}(k) &= (k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi(k) \\ g^{\mu\nu}\Pi_{\mu\nu}(k) &= 3k^2\Pi(k). \end{aligned} \tag{28.131}$$

Hence,

$$\begin{aligned} \Pi(k) &= -\frac{g^2}{3k^2}\sum_{n=-\infty}^{\infty}\int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\left(\gamma_\mu \frac{1}{\not q - m_n}\gamma^\mu \frac{1}{\not k + \not q - m_n}\right) \\ &= -\frac{g^2}{3k^2}\sum_{n=-\infty}^{\infty}\int \frac{d^4q}{(2\pi)^4} \frac{\mathrm{tr}[\gamma_\mu(\not q + m_n)\gamma^\mu(\not k + \not q - m_n)]}{(q^2 - m_n^2)[(k+q)^2 - m_n^2]} \\ &= -\frac{g^2}{3k^2}\sum_{n=-\infty}^{\infty}\int \frac{d^4q}{(2\pi)^4} \frac{\mathrm{tr}[(-2\not q + 4m_n)(\not k + \not q - m_n)]}{(q^2 - m_n^2)[(k+q)^2 - m_n^2]} \\ &= -\frac{8g^2}{3k^2}\sum_{n=-\infty}^{\infty}\int \frac{d^4q}{(2\pi)^4} \frac{-q\cdot(k+q) + 2m^2}{(q^2 - m_n^2)[(k+q)^2 - m_n^2]}, \end{aligned} \tag{28.132}$$

Using Feynman parameter to combine the denominator, we have

$$\begin{aligned} \frac{1}{(q^2 - m_n^2)[(k+q)^2 - m_n^2]} &= \int_0^1 dx \frac{1}{[x[(k+q)^2 - m_n^2] + (1-x)(q^2 - m_n^2)]^2} \\ &= \int_0^1 dx \frac{1}{[q^2 + 2kqx + k^2x - m_n^2]^2} \\ &= \int_0^1 dx \frac{1}{[l^2 + x(1-x)k^2 - m_n^2]^2}, \end{aligned} \tag{28.133}$$

where $l = q + kx$. With this transformation, the numerator becomes

$$-q \cdot (k+q) + 2m^2 = -l^2 + x(1-x)k^2 + 2m_n^2\,, \tag{28.134}$$

where I have dropped terms linear in l because they vanish when taking the l integral. Relabeling l as q we have

$$\Pi(k) = -\frac{8g^2}{3k^2}\sum_{n=-\infty}^{\infty}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{-q^2 + x(1-x)k^2 + 2m_n^2}{[q^2 + x(1-x)k^2 - m_n^2]^2}\,, \tag{28.135}$$

Rotate to Euclidean momenta, $q \to iq_E$, we have

$$\Pi(k) = -\frac{8g^2}{3k^2}\sum_{n=-\infty}^{\infty}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 q_E}{(2\pi)^4}\frac{q^2 - x(1-x)k^2 + 2m_n^2}{[q^2 + x(1-x)k^2 + m_n^2]^2}\,, \tag{28.136}$$

Using the Schwinger proper-time parameter,

$$\int_0^\infty \mathrm{d}t\;\; te^{-At} = -\frac{\mathrm{d}}{\mathrm{d}A}\int_0^\infty \mathrm{d}te^{-At} = -\frac{\mathrm{d}}{\mathrm{d}A}\frac{1}{A} = \frac{1}{A^2}\,, \tag{28.137}$$

we have

$$\begin{aligned}\Pi(k) = -\frac{8g^2}{3k^2}\sum_{n=-\infty}^{\infty}\int_0^1 \mathrm{d}x \int_0^\infty \mathrm{d}t\; t \int \frac{\mathrm{d}^4 q_E}{(2\pi)^4}[q^2 - x(1-x)k^2 + 2m_n^2] \\ \exp\left\{-t[q^2 + x(1-x)k^2 + m_n^2]\right\}\,.\end{aligned} \tag{28.138}$$

Using the following identities

$$\begin{aligned}\int \frac{\mathrm{d}^4 q_E}{(2\pi)^4}e^{-tq_E^2} &= \frac{1}{16\pi^4}\int d^3\Omega_E \int_0^\infty \mathrm{d}q_E\;\; q_E^3 e^{-tq_E^2} = \frac{2\pi^2}{16\pi^4}\int_0^\infty \mathrm{d}q_E\;\; q_E^3 e^{-tq_E^2} \\ &= \frac{\pi^2}{16\pi^4}\int_0^\infty \mathrm{d}q_E^2\;\; q_E^2 e^{-tq_E^2} = \frac{\pi^2}{16\pi^4}\frac{1}{t^2} = \frac{1}{16\pi^2 t^2}\,,\end{aligned} \tag{28.139}$$

and

$$\int \frac{\mathrm{d}^4 q_E}{(2\pi)^4}q_E^2 e^{-tq_E^2} = -\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\mathrm{d}^4 q_E}{(2\pi)^4}e^{-tq_E^2} = \frac{1}{8\pi^2 t^3}\,, \tag{28.140}$$

the loop integral becomes

$$\Pi(k) = -\frac{g^2}{6\pi^2 k^2} \sum_{n=-\infty}^{\infty} \int_0^1 \mathrm{d}x \int_0^{\infty} \frac{\mathrm{d}t}{t} \left[\frac{2}{t} - x(1-x)k^2 + 2m_n^2 \right]$$
$$\exp\left\{ -t[x(1-x)k^2 + m_n^2] \right\} . \tag{28.141}$$

Integrate the first term by parts, we have

$$\int_0^{\infty} \frac{\mathrm{d}t}{t} \ \frac{2}{t} \exp\left\{ -t[x(1-x)k^2 + m_n^2] \right\} \tag{28.142}$$
$$= -\frac{2}{t} \exp\left\{ -t[x(1-x)k^2 + m_n^2] \right\} \bigg|_0^{\infty} - \int_0^{\infty} \frac{\mathrm{d}t}{t} \ 2\left[x(1-x)k^2 + m_n^2 \right]$$
$$\exp\left\{ -t[x(1-x)k^2 + m_n^2] \right\} . \tag{28.143}$$

Ignoring the boundary term, which will be elaborate on later, (28.141) becomes

$$\Pi(k) = \frac{g^2}{2\pi^2} \sum_{n=-\infty}^{\infty} \int_0^1 \mathrm{d}x \ \ x(1-x) \int_0^{\infty} \frac{\mathrm{d}t}{t} \exp\left\{ -t[x(1-x)k^2 + m_n^2] \right\} . \tag{28.144}$$

Using Jocobi ϑ_3 function

$$\vartheta_3(\tau) = \sum_{n=-\infty}^{\infty} \exp\{ i\pi\tau n^2 \} , \tag{28.145}$$

and taking $m_n^2 = \frac{n^2}{R^2}$ we have

$$\Pi(k) = \frac{g^2}{2\pi^2} \int_0^1 \mathrm{d}x \ \ x(1-x) \int_0^{\infty} \frac{\mathrm{d}t}{t} \ \ e^{-t[x(1-x)k^2]^2} \vartheta_3\left(\frac{it}{\pi R^2} \right) . \tag{28.146}$$

For $k = 0$, we have

$$\Pi(0) = \frac{g^2}{12\pi^2} \int_0^{\infty} \frac{\mathrm{d}t}{t} \ \ \vartheta_3\left(\frac{it}{\pi R^2} \right) . \tag{28.147}$$

Since this integral is UV and IR divergent, we introduce both UV and IR cutoffs

$$\Pi(0) = \frac{g^2}{12\pi^2} \int_{r\Lambda^{-2}}^{r\mu_0^{-2}} \frac{\mathrm{d}t}{t} \ \ \vartheta_3\left(\frac{it}{\pi R^2} \right) = \frac{g^2 b}{16\pi^2} \int_{r\Lambda^{-2}}^{r\mu_0^{-2}} \frac{\mathrm{d}t}{t} \ \ \vartheta_3\left(\frac{it}{\pi R^2} \right) , \tag{28.148}$$

where $b = 4/3$ is the beta-function coefficient of a single Dirac fermion and $r = \pi/4$ is a numerical factor.

Now, consider the boundary term in (28.143). The boundary term with the cutoff becomes

$$\frac{g^2}{6\pi^2 k^2}\frac{2}{t}e^{-t[x(1-x)k^2+m_n^2]}\bigg|_{r\Lambda^{-2}}^{\infty} = -\frac{g^2}{6\pi^2 k^2}\frac{2\Lambda^2}{r}e^{-r[x(1-x)k^2+m_n^2]/\Lambda^2}\,. \tag{28.149}$$

This term is quadratically divergent. It leads to a term in the vacuum polarization tensor of the form

$$\Pi_{\mu\nu}(k) \propto g_{\mu\nu}\, g^2\Lambda^2 \tag{28.150}$$

which corresponds to a mass term for the gauge boson. It is a signal that our regularization scheme is not gauge invariant. In any gauge invariant regularization scheme, such as dimensional regularization, this quadratic divergence is absent. Therefore we can safely ignore it here. On the other hand, the second term in (28.143) is logarithmic divergent.

The one-loop-corrected gauge coupling is then given by

$$\begin{aligned}
g(\Lambda) &= \left(\frac{1}{1-\Pi(0)}\right)^{1/2} g(\mu_0) \\
\alpha^{-1}(\Lambda) &= [1-\Pi(0)]\alpha^{-1}(\mu_0) = \alpha^{-1}(\mu_0) - \frac{4\pi}{g^2}\Pi(0) \\
\alpha^{-1}(\Lambda) &= \alpha^{-1}(\mu_0) - \frac{b}{4\pi}\int_{r\Lambda^{-2}}^{r\mu_0^{-2}} \frac{\mathrm{d}t}{t}\ \vartheta_3\left(\frac{it}{\pi R^2}\right)\,.
\end{aligned} \tag{28.151}$$

28.15 Solution to Problem 13

In general, the gauge bosons which comprise the local GUT satisfy

$$\mathbf{P}\cdot\mathbf{X} = 0 \bmod 1, \tag{28.152}$$

where $\mathbf{X}$, in our case, is defined by

$$\mathbf{X} = \mathbf{V}_6 + 0\times\mathbf{W}_3 + 0\times\mathbf{W}_2 = \mathbf{V}_6. \tag{28.153}$$

Clearly, the E_8 roots which survive the projection are

$$(000\underline{\pm1\pm1000}), \quad \pm(\underline{01-1}00000). \tag{28.154}$$

In order to find the gauge group, we should find a suitable basis and calculate the Cartan Matrix. We choose

$$\begin{array}{lllllllll}
\boldsymbol{\alpha}_1 = 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
\boldsymbol{\alpha}_2 = 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
\boldsymbol{\alpha}_3 = 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
\boldsymbol{\alpha}_4 = 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
\boldsymbol{\alpha}_5 = 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\boldsymbol{\alpha}_6 = 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\boldsymbol{\alpha}_7 = 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0
\end{array} \tag{28.155}$$

We can again compute the Cartan Matrix of the 7 roots and find:

$$\mathbb{A} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{28.156}$$

The Cartan Matrix for SO(10) is

$$\mathbb{A}_{SO(10)} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}, \tag{28.157}$$

so it seems that our local GUT is precisely $SO(10) \times SU(3)$.

In order to show that there's a spinor (**16**) of SO(10), we must find the right mover and the left mover which satisfy the masslessness conditions, Eqs. (20.147) and (20.148). We find

$$a_R^1 = \frac{1}{2} - \frac{1}{2}\left\{\frac{5}{36} + \frac{2}{9} + \frac{1}{4}\right\} = -\frac{7}{36}. \tag{28.158}$$

This gives us the mass equation

$$|\mathbf{r} + \mathbf{v}_6|^2 = \frac{7}{18} \tag{28.159}$$

Now, consider the SO(8)*vector* $|0010\rangle$ or spinor $\left|-\frac{1}{2}-\frac{1}{2}\frac{1}{2}\frac{1}{2}\right\rangle$. They both satisfy the mass relationship:

$$|\mathbf{r}+\mathbf{v}_6|^2=\frac{7}{18}. \tag{28.160}$$

Next, we calculate the mass equation in the left moving sector:

$$a_L^1=1-\frac{1}{2}\left(\frac{5}{36}+\frac{2}{9}+\frac{1}{4}\right)=\frac{25}{36} \tag{28.161}$$

Showing that there is an SO(10) spinor is as easy as finding some E_8 lattice vector that is all $\pm\frac{1}{2}$s and which obeys the masslessness condition

$$|\mathbf{P}+\mathbf{V}_6|^2=\frac{50}{36} \tag{28.162}$$

We find that

$$P \;=\; \frac{1}{2}(-1,-1,-1,\pm1,\pm1,\pm1,\pm1,\pm1)\,(0^8) \tag{28.163}$$

with an even number of minus signs works. These are just the 16 states in the spinor representation of $SO(10)$.

Finally we can show how the projection condition works on the left mover:

$$(2\mathbf{P}+\mathbf{V}_6)\cdot\mathbf{V}_6=2\mathbf{P}\cdot\mathbf{V}_6+\mathbf{V}_6^2=-1+\frac{7}{18}. \tag{28.164}$$

This is fortunate, because the right mover transforms as

$$(2\mathbf{r}+\mathbf{v}_6)\cdot\mathbf{v}_6=2\mathbf{r}\cdot\mathbf{v}_6+\mathbf{v}_6^2=-1+\frac{7}{18}. \tag{28.165}$$

We have no oscillators so $\phi=1$, and $\gamma=1$ in the first twisted sector. This means that the GSO projection acts on the states as

$$\Delta=\gamma\phi\;\exp\{i\pi\,[(2\mathbf{P}+\mathbf{V}_6)\cdot\mathbf{V}_6-(2\mathbf{r}+\mathbf{v}_6)\cdot\mathbf{v}_6]\}=1. \tag{28.166}$$

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